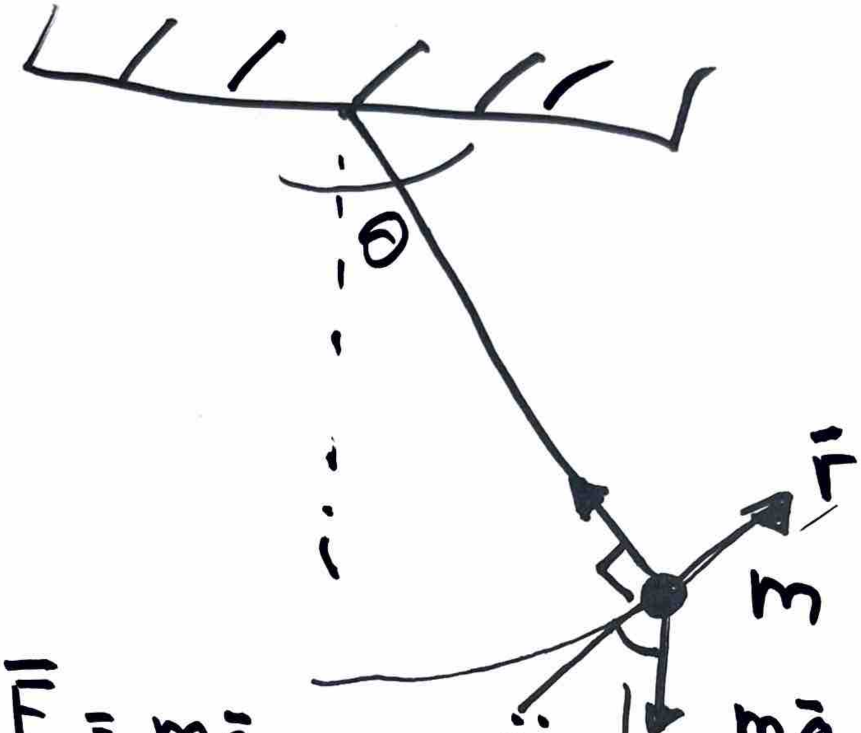
1-



$$\bar{F} = m\bar{a} = m\ddot{\bar{x}}$$

$$\left(\frac{\pi}{2} - \theta\right) = \alpha$$

$$\bar{F} = m\bar{g} \Rightarrow$$

$$-mg\cos\alpha = -mg\sin\theta = ma = mL\ddot{\theta}$$

$$mL\ddot{\theta}(t) + mg\sin\theta(t) = 0$$

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

$$\sin\theta \approx \theta$$

$$\ddot{\theta} + \frac{g}{L}\theta = 0$$

-2-

$$\ddot{\theta} + \frac{g}{L}\sin\theta = \emptyset$$

$$t = T\tau$$

t, τ independent variables

$[t] = \text{Time}$

T is a const

$[T] = \text{Time}$

$[\tau] = 1$

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \frac{1}{T}\frac{d}{d\tau}$$

$$\boxed{T^2 = \frac{L}{g}}$$

$$\frac{d^2}{dt^2} = \frac{1}{T^2}\frac{d^2}{d\tau^2}$$

$$\frac{d^2}{d\tau^2}\theta(\tau) + \frac{gT^2}{L}\sin\theta(\tau) = \emptyset$$

$$\left[\frac{d^2}{d\tau^2}\theta(\tau) + \sin\theta(\tau) = 0\right]\frac{d\theta}{d\tau}$$

$$\begin{cases} \dfrac{d}{d\tau}\theta(\tau) = \omega(\tau) \\[2ex] \dfrac{d}{d\tau}\omega(\tau) = -\sin\theta(\tau) \end{cases}$$

$$\ddot{\theta}\,\dot{\theta} + \dot{\theta}\sin\theta = 0$$

$$\left(\frac{d^2}{d\tau^2}\theta(\tau)\right)\frac{d}{d\tau}\theta(\tau) = \frac{1}{2}\frac{d}{d\tau}\left(\theta(\tau)\right)^2$$

$$\ddot{\theta}\,\dot{\theta} = \frac{d}{d\tau}\frac{\dot{\theta}^2}{2}$$

$$\dot{\theta}\sin\theta = -\frac{d}{d\tau}\cos(\theta)$$

$$\longrightarrow -\frac{d}{dT}\cos\theta$$

$$\underbrace{\dot{\theta}\,\ddot{\theta}}_{} + \dot{\theta}\sin\theta = 0$$

$$L = \frac{d}{dT}\frac{\dot{\theta}^2}{2}$$

$$\frac{d}{dT}\frac{\dot{\theta}^2}{2} - \frac{d}{dT}\cos\theta = 0$$

$$\frac{d}{dT}\left[\frac{\dot{\theta}^2}{2} - \cos\theta\right] = 0$$

$$\frac{\dot{\theta}^2}{2} - \cos\theta = E$$

$$\frac{d}{dT}\theta(T) = \sqrt{2(E + \cos\theta)}$$

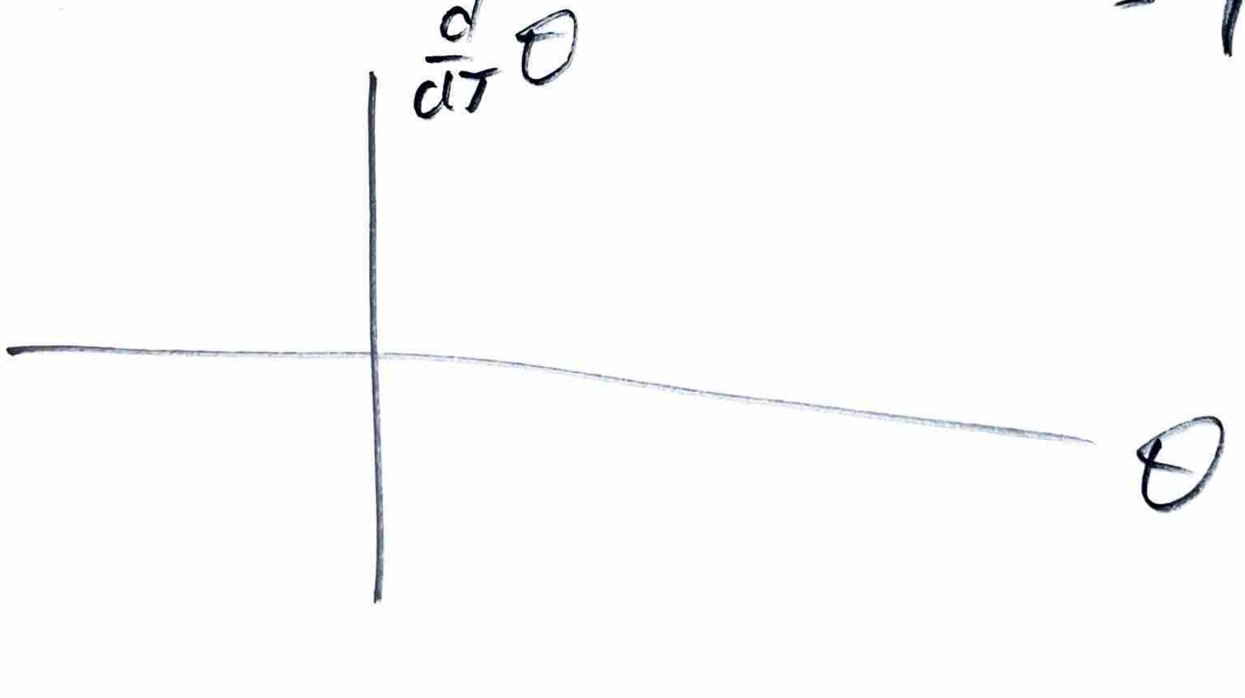
$$\frac{d}{d\tau}\theta(\tau) = \sqrt{2(E + \cos\theta)}$$

IF $|E| < 1$

$$\cos\theta_{max} = -E$$

$$\theta_{max} = \mathrm{ArcCos}\, E$$

IF $|E| > 1$, any θ is possible



Flow on a LINE

$$\frac{d}{dt} x(t) = f(x(t))$$ - autonomous

Nonlinear, first order ODE

Most general form

$$\frac{d}{dt} x(t) = f(x(t), t)$$

non-autonomous

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$$\dot{x}(t) = \sin x(t)$$

$$\frac{dx}{dt} = \sin x$$

$$\int \frac{dx}{\sin x} = \int dt$$

$$\int \frac{dx}{\sin x} = \ln\left(\tan\left(\frac{x}{2}\right)\right)$$

$$\frac{d}{dx}\left[\ln\left(\tan(x/2)\right)\right] = \frac{1}{2\tan(x/2)\cos^2(x/2)} =$$

$$= \frac{1}{2\,\frac{\sin(x/2)}{\cos(x/2)}\cos^2(x/2)}$$

$$= \frac{1}{2\sin(x/2)\cos(x/2)} = \frac{1}{\sin x}$$

$$t+c = \ln(\tan(x/2))$$

$$e^{t+c} = \tan x/2$$

$$2 \, \mathrm{ArcTan}(e^{t+c}) = x(t)$$

$$T = \sqrt{\frac{L}{g}}$$

① Let $x(t=0) = \pi/2$

$x(t) = ?$

② what happens with $x(t)$ as $t \to \infty$

③ Types of trajectories

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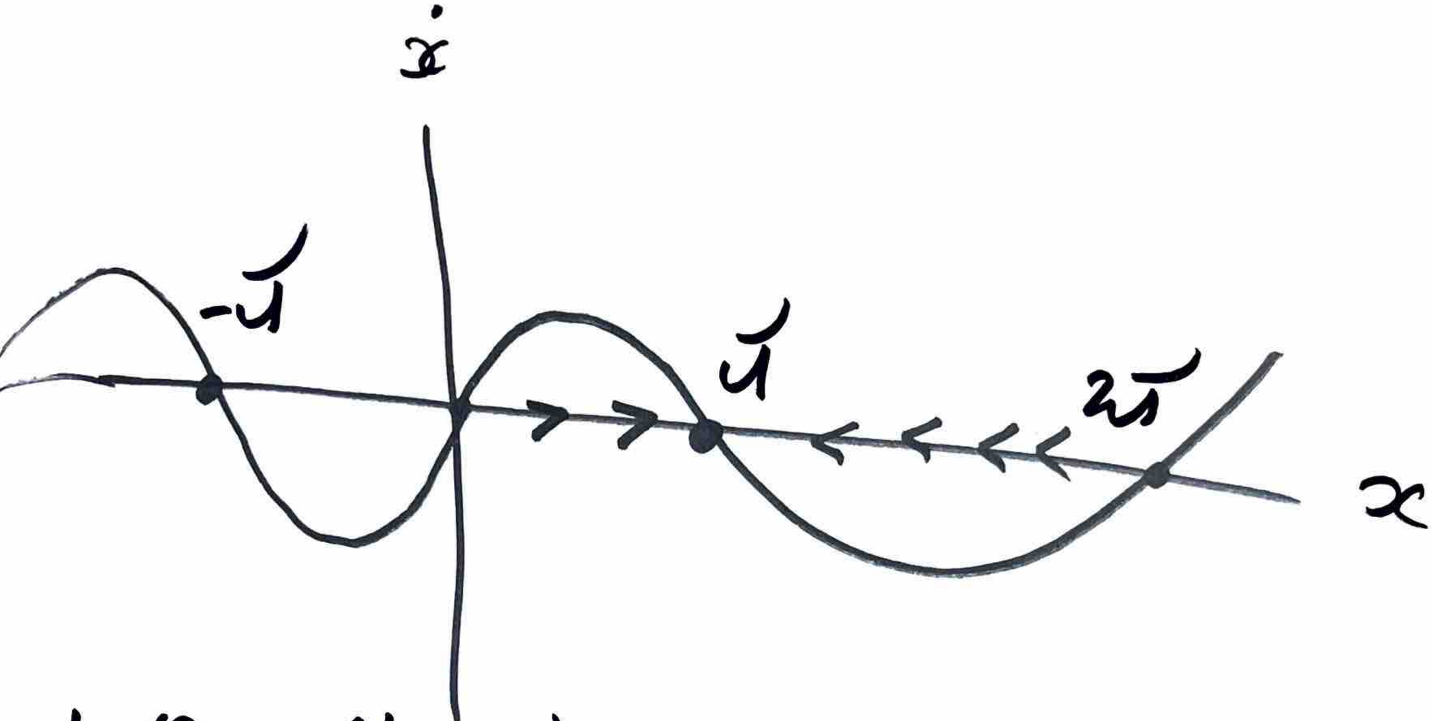
$\dot{x} = \sin x$



• Let [illegible]

$0 < x(t=0) < \pi$

$\dot{x}(t=0) > 0$

$x(t)$ grows until $x = \pi$, stay at π

• Let $\pi < x(t=0) < 2\pi$

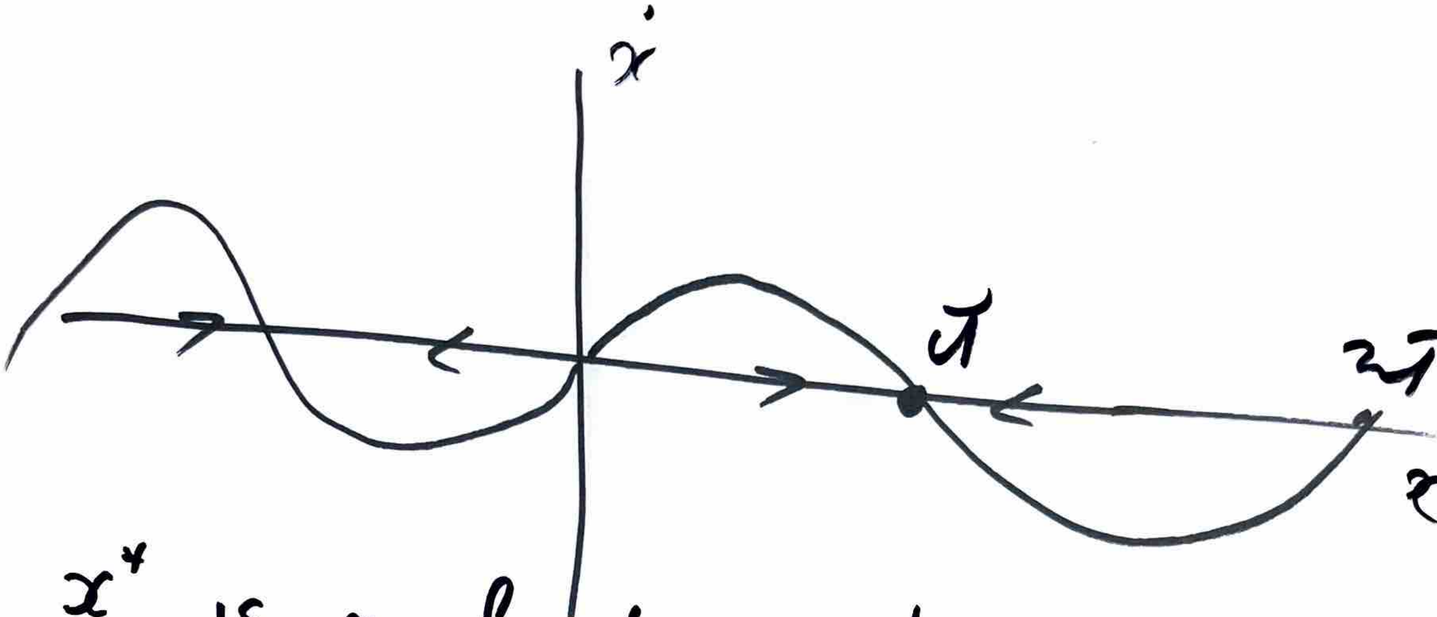
$\dot{x}(t \neq 0) < 0$

$x(t)$ will decrease

untill $x(t) = \pi$, stay there

- $x(t=0) = 0 \Rightarrow x(t) \equiv 0$
- $x(t=0) = \pi$. $x(t) \equiv \pi$
- $x(t=0) = m\pi$, $x(t) \equiv \pi \cdot m$

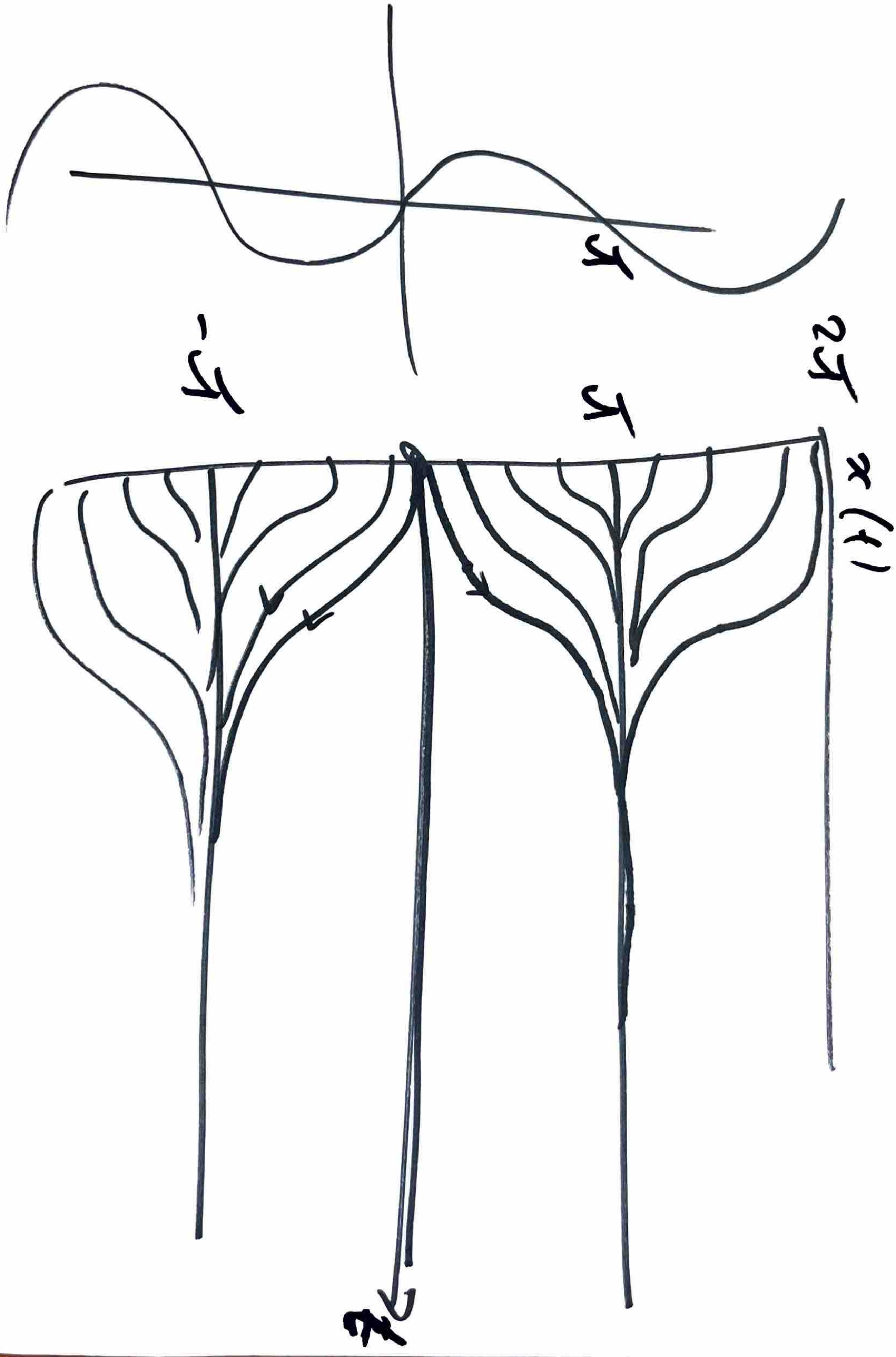
  $m = 0, 1, 2, \ldots$



x^* is a fixed point

if $f(x^*) = \emptyset$

For this example $x^* = m\pi$; $m = \emptyset, \pm 1, \pm 2 \ldots$


π
-π
π
2π
x(t)
t

To solve $\dot{x}(t) = f(x(t))$

① Find all fixed points:

Find x^*

such that $f(x^*) = \emptyset$

② Analyze regions between fixed points:

IF $f(x) > \emptyset$ => flow to the right

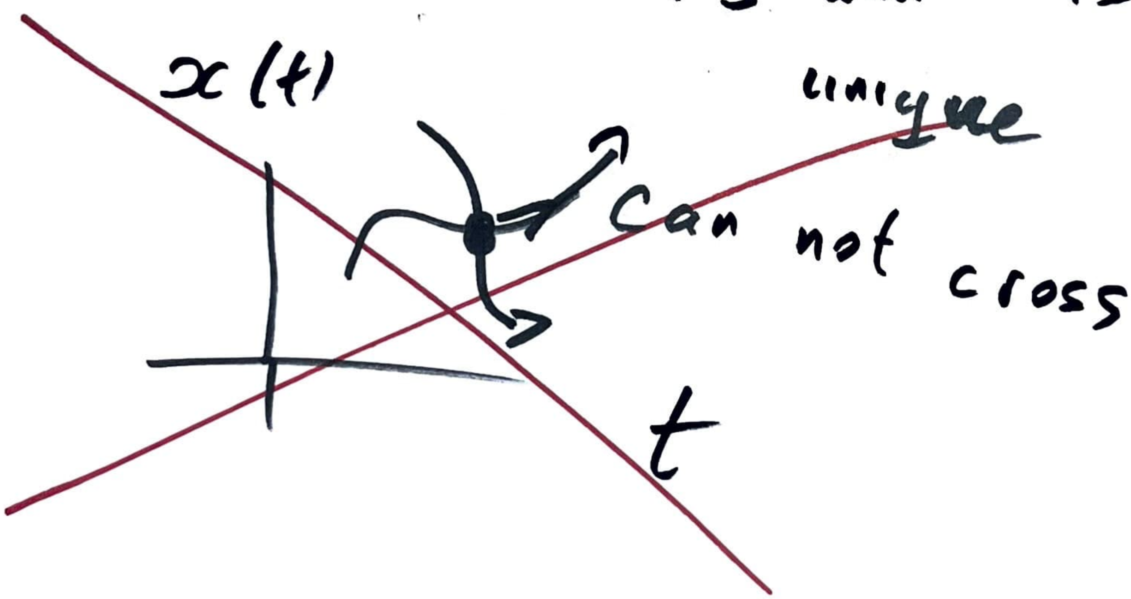
—"— $f(x) < \emptyset$ —"—"—— left

—"— $f(x) = \emptyset$ stay where you are

Existence and uniqueness

if $f(x)$ is C', continiously differentiable

$\Rightarrow$ $x(t)$ exists and is unique



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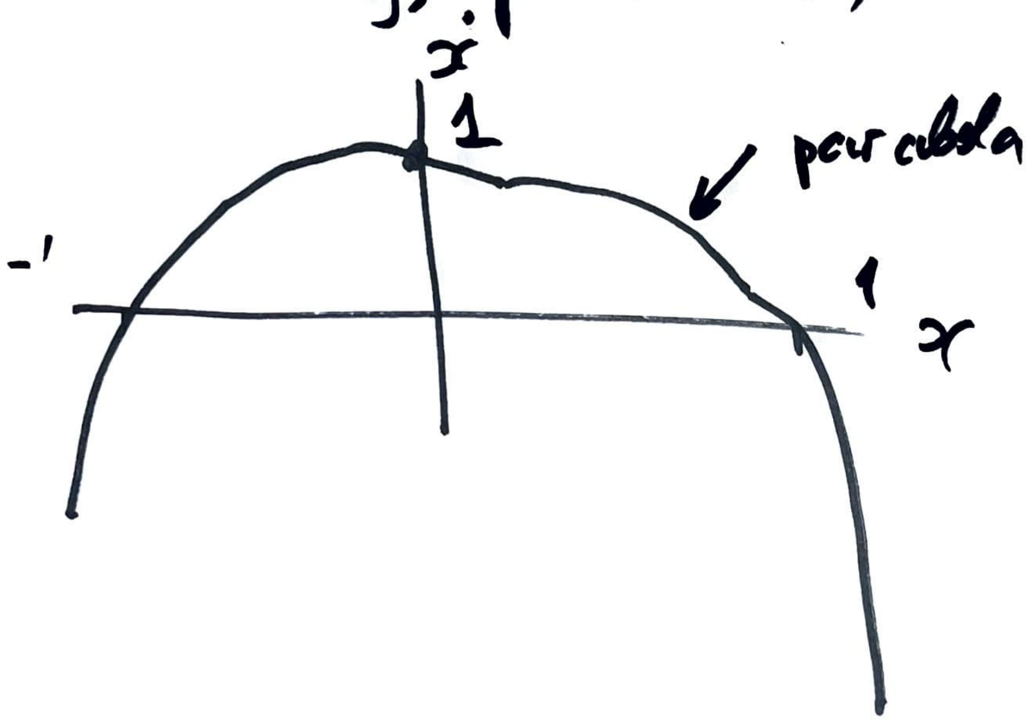
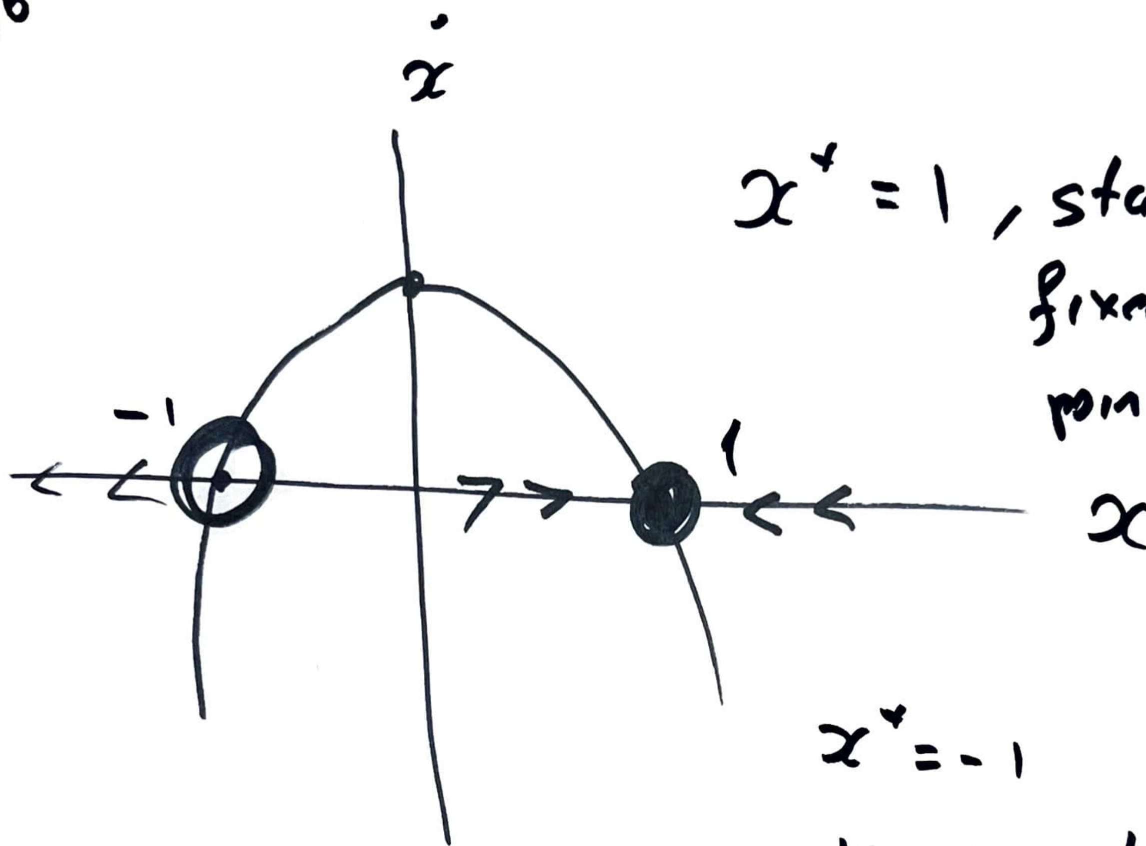
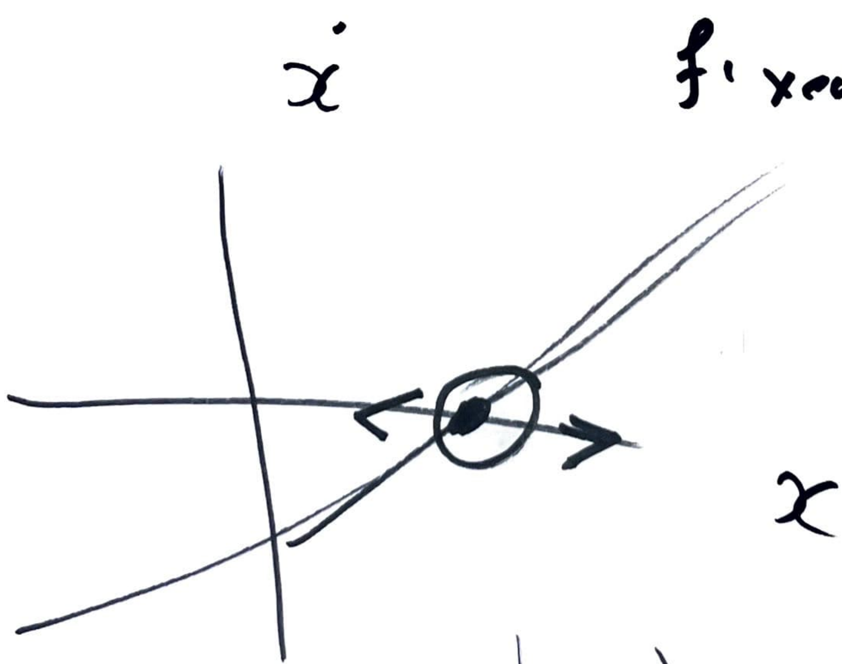
Study $\dot{x}(t) = 1 - x^2(t)$
find fixed points analyze
stability, plot $x(t)$

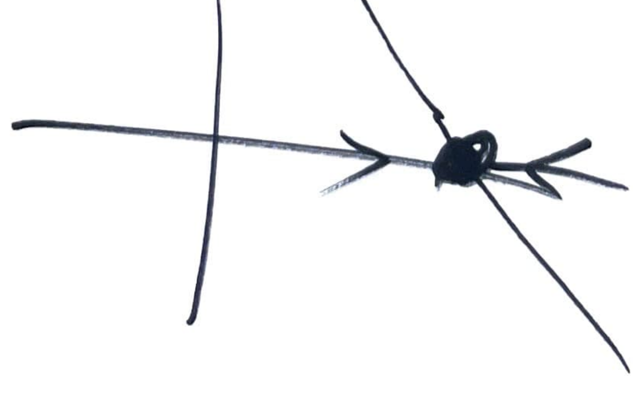




$x^* = 1$, stable fixed point

$x^* = -1$ is unstable fixed point;





$$\dot{x} = 1 - x^2 = \frac{dx}{dt}$$

$$\int \frac{dx}{1-x^2} = \text{[illegible]}$$

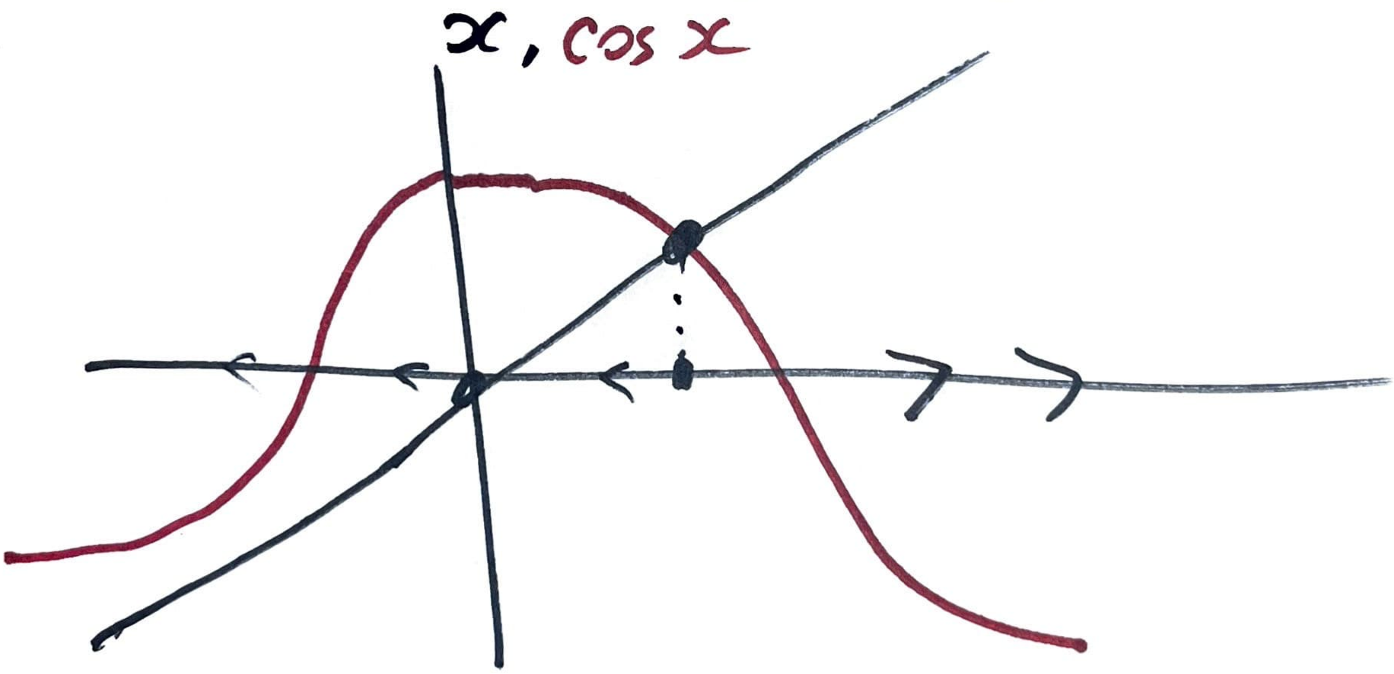
$$= \text{Arc Tan } x = t + c$$

$$x = \tan(t+c)$$

18 Example [same steps]

$$\dot{x} = x - \cos x$$

Find fixed points, analyze stability, find trajectories.



$x^* = \cos x^*$ – does not have analytical sltn

x^* is crossing of $x, \cos x$