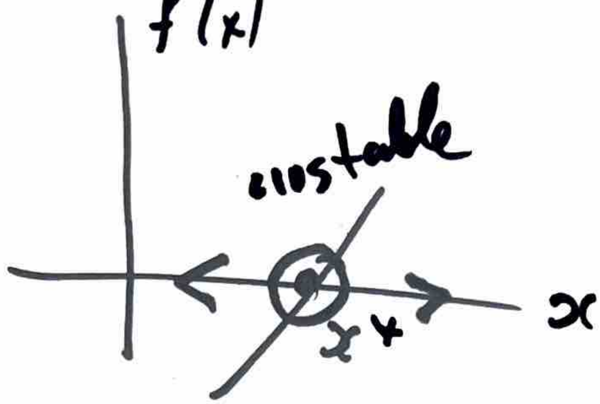
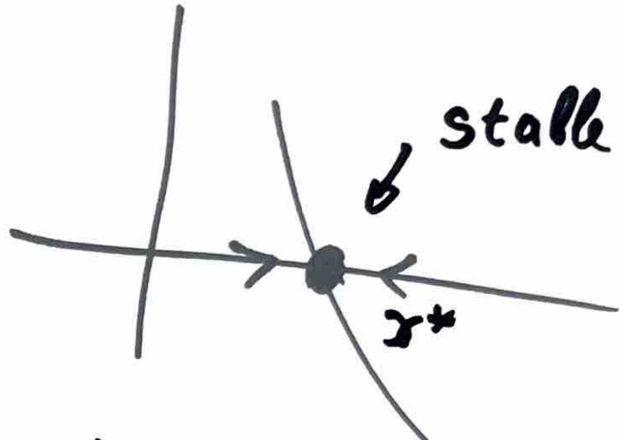
$$\dot{x} = f(x(t))$$

yurilvov.com/ODE/index.html





Linear Stability Analyses

Let x^* be such $f(x^*) = 0$.

$$x(t) = x^* + \rho(t)$$

$$\dot{x}(t) = \dot{\rho}(t) \qquad |\rho(t)| \ll 1$$

$$\dot{\rho}(t) = f(x^* + \rho(t)) \approx$$

$$\approx \underbrace{f(x^*)}_{=0} + f'(x^*)\rho(t) + \underbrace{f''(x^*)\rho^2(t)/2}_{\text{small}} + \dots$$

$$f'(x^*) \neq 0, \quad f''(x^*)\rho^2(t) \ll f'(x^*)\rho(t)$$

$\varnothing$

$$\dot{g}(t) = f'(x^*)\, g(t)$$

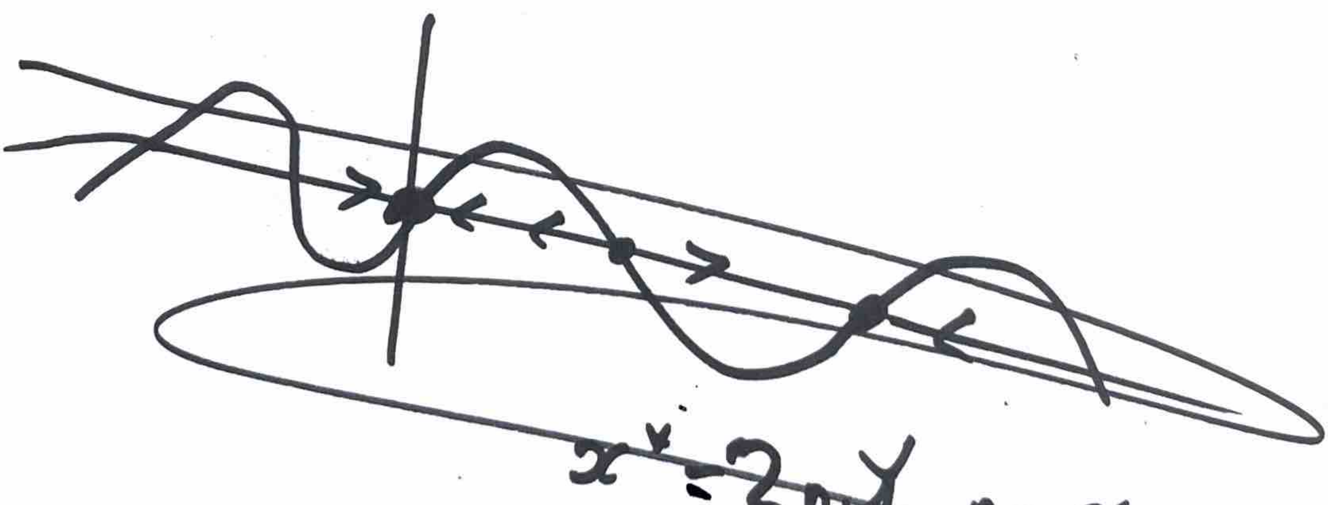
$$g(t) = g(t=0)\, e^{f'(x^*)\cdot t}$$

IF $f'(x^*) > 0$, then $\lim_{t \to \infty} g(t) = \infty$

$f'(x^*) < 0$, then $\lim_{t \to \infty} g(t) = 0$

zero

$$\dot{x} = \sin x$$



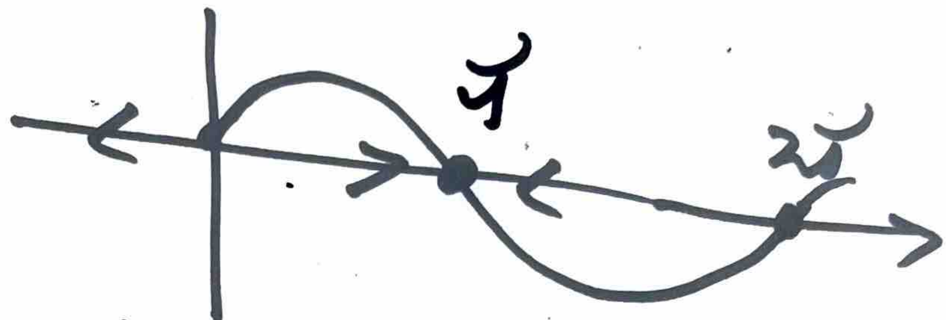
$x^* = 2n\pi$, $n = 0, 1, 2, 3,$

$-1, -2, -3$

stable

3

$\dot{x} = \sin x$



$n = 0, \pm 1, \pm 2 \ldots$

$x^* = 2\pi n$ unstable

$x^* = (2n+1)\pi$ stable

IF $x^* = 2\pi n$

$f'(x^*) = \cos(2\pi n) = \cos(0) = 1$

$f'(x^*) = 1 > 0$ unstable

$x^* = (2n+1)\pi$

$f'(x^*) = \cos(\pi(2n+1))$

$= \cos\pi = -1 < 0$

stable

-4-

Let $f'(x^*) = 0$, $f''(x^*) \neq 0$

$$\dot{g}(t) = \frac{f''(x^*)}{2} \cdot g^2(t) \; ; \; \alpha \equiv f''(x^*)/2$$

$$\frac{dg}{dt} = \alpha g^2$$

$$-\frac{1}{g} = \int \frac{dg}{g^2} = \int \alpha \, dt = \alpha t + c$$

$$g(t) = \frac{1}{-c - \alpha t}$$

$$g(t=0) = 1/A, \quad A > 0$$

$t \to \infty$

$$g(t) = -\frac{1}{A + \alpha t}, \quad \text{IF } \alpha > 0 \quad g(t) \to 0$$

$$\lim_{t \to \infty} g(t) = 0, \quad \alpha > 0 \qquad \text{IF } \alpha < 0, \; g(t)$$

$$\lim_{t \to \infty} g(t) = -\infty \quad \alpha < 0$$

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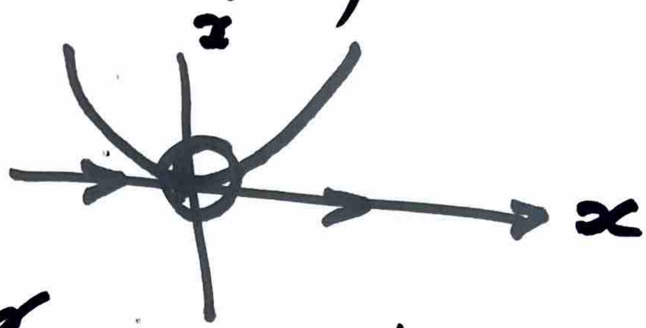
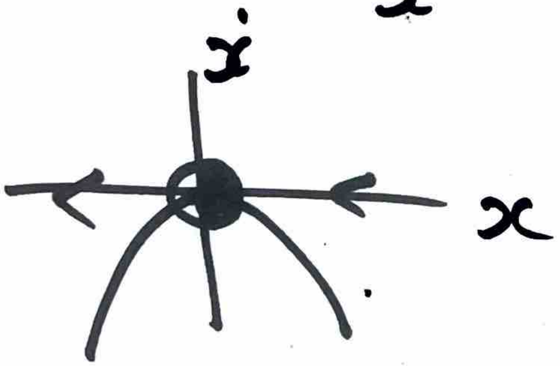
- $\dot{x}(t) = x^2(t)$



$x = \emptyset$ - semistable: stable $x^* = \emptyset^-$

SEMISTABLE unstable $x^* = \emptyset^+$

$\dot{x} = -x^2$



unstable $x^* = \emptyset^-$

stable $x^* = \emptyset^+$

$$f(x^*) = f'(x^*) = f''(x^*) = 0$$

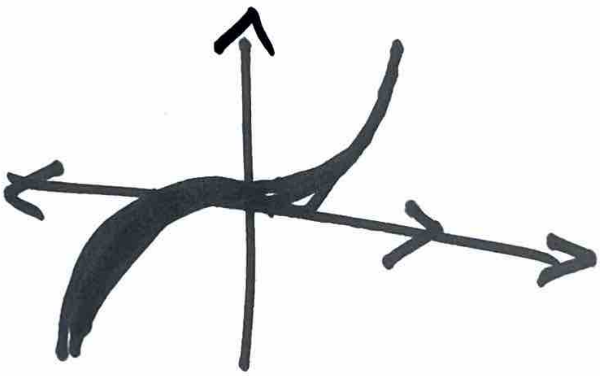
$$f'''(x^*) \neq 0$$

$$\dot{\eta}(t) = f'''(x^*)\,\eta^3(t)/6$$

$$\dot{x} = x^3$$

$x^* = 0$

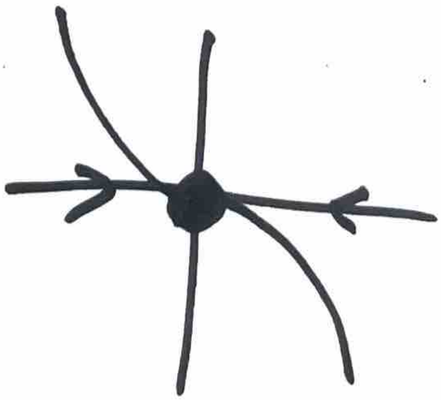
is unstable



$$\dot{x} = -x^3$$

$x^* = 0$

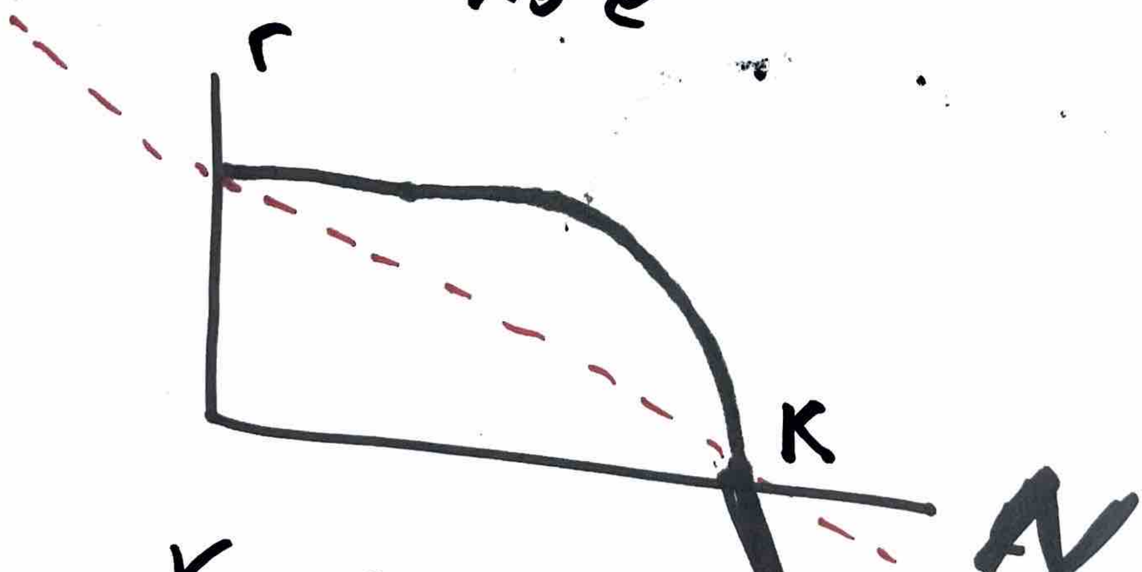
stable



Simplest population growth model

$$\frac{d}{dt} N(t) = r\,N(t), \quad r > 0$$

$$N(t) = N_0 e^{rt}$$



K - carrying capacity

$$\frac{d}{dt} N(t) = r\,N(t)\left(1 - \frac{N(t)}{K}\right)$$

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$$\frac{d}{dt} N(t) = r\, N(t)\left(1 - \frac{N(t)}{K}\right)$$

$$\int \frac{d\,N(t)}{N(t)\left(1 - \frac{N(t)}{K}\right)} = r\int dt = rt + C$$

$$\cancel{\frac{1}{N\left(1-\frac{N}{K}\right)}} =$$

$$A = 1, \quad B = 1/K$$

$$\frac{1}{N\left(1 - \frac{N}{K}\right)} = \frac{A}{N} + \frac{B}{1 - \frac{N}{K}}$$

$$= \frac{A\left(1 - \frac{N}{K}\right) + BN}{N\left(1 - N/K\right)} =$$

$$= \frac{A + N(B - 1/K)}{N(1 - N/K)}$$

$$\int \frac{dN}{N(1-N/K)} = \int dN\left(\frac{1}{N} + \frac{1/K}{1-N/K}\right)$$

$$= \log N - \log(1-N/K)$$

$$= \log\frac{N}{1-N/K}$$

$$\left[\log(1-N/K)\right]' = \frac{-1/K}{1-N/K}$$

$$\frac{N}{1-N/K} = e^{rt+c}$$

$$e \equiv e^{rt+c}$$

$$\cancel{N \cdot e = e - N/K \cdot e}$$

$$\cancel{Ne(1+1/K) = e}$$

$$\cancel{N(t)}$$

$$N(t) = e^{rt+c} - \frac{N(t)}{K} e^{rt+c}$$

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$$N(t)\left(1 + \frac{1}{K} e^{rt+c}\right) = e^{rt+c}$$

$$N(t) = \frac{e^{rt+c}}{\frac{1}{K} e^{rt+c} + 1}$$

$$N(t) = \frac{1}{1/K + e^{-rt+c'}}$$

$$N(t=0) = \frac{1}{1/K + C''} = N_0$$

$$1 = N_0\left(1/K + C''\right)$$

$$N(t) = \frac{N_0 K}{N_0 + e^{-rt}(K - N_0)}$$

$$1/N_0 - 1/K = C''$$

$$N(t) = \frac{1}{1/K + e^{-rt}\left(\frac{1}{N_0} - \frac{1}{K}\right)}$$

$$N(t) = \frac{K}{1 + e^{-rt}\left(\frac{K}{N_0} - 1\right)}$$

$\underline{r > 0}$

$\dot{N} = rN(1 - N/K)$

Fixed points $N = 0$

$N = K$

$\dot{N} = f(N)$

$f(N) = rN - \frac{r}{K}N^2$

$f'(N) = r - \frac{2rN}{K}$

$N = 0, f'(N) = r \rightarrow$ unstable

$N = K, f'(N) = r - 2r = -r < 0$ stable

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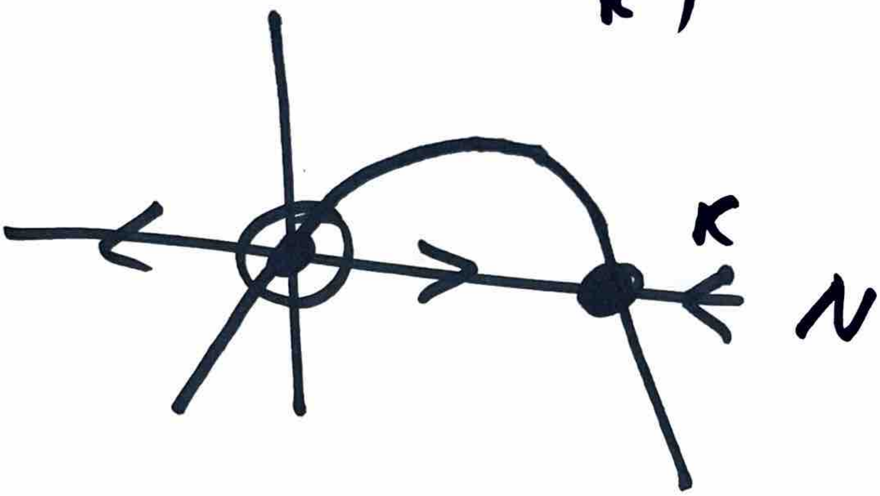
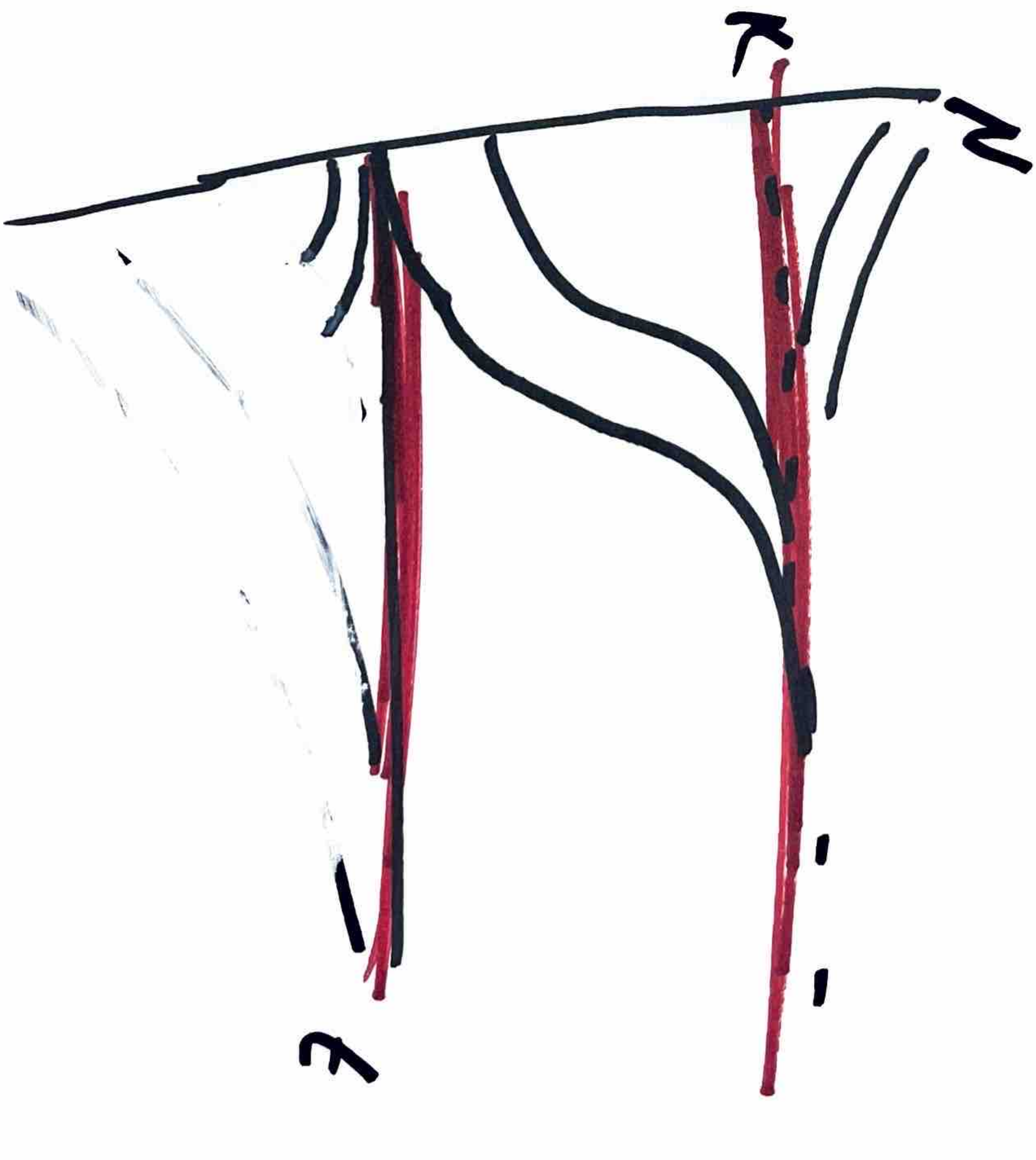
$$\dot{N} = rN\left(1-\frac{N}{K}\right)$$



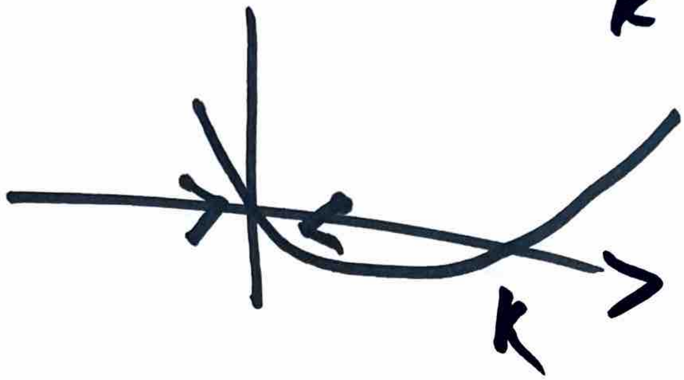
$N=0$ unstable

$N=K$ stable



$r > 0$

$$\dot{N} = -rN\left(1 - \frac{N}{K}\right)$$



K

threshold

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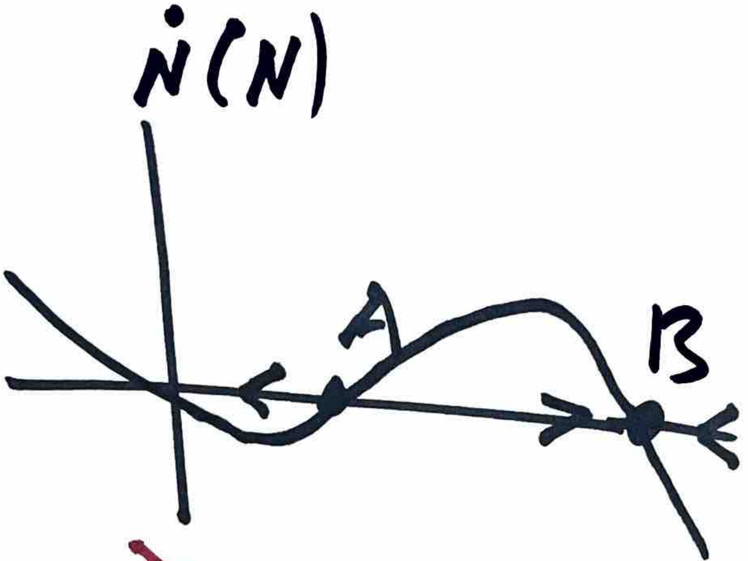
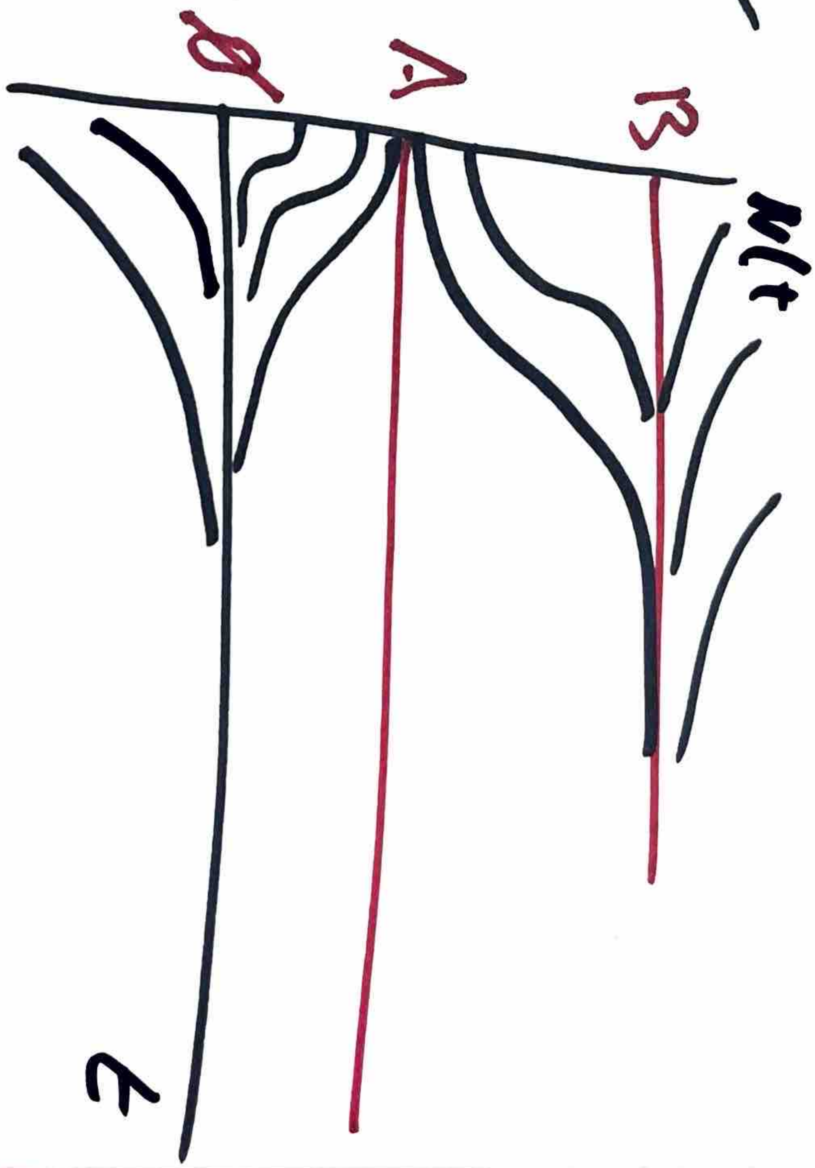
$$\dot{N} = -rN\left(1-\frac{N}{A}\right)\left(1-\frac{N}{B}\right)$$



$B > A > 0$

N



A - threshold

B - carrying capacity

Theorem of existence and uniqueness:

$$\left.\begin{aligned} \dot{x}(t) &= f(x(t)), \\ x(t=t_0) &= x_0 \end{aligned}\right\} (*) \text{ IVP}$$

Let $f(x)$ and $f'(x)$ exist and are continious in some open interval around x_0

Then solution to $(*)$ IVP

$\exists!$ ← unique in some

exists

interval to

IVP ≡ initial value problem

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$\dot{x} = \sqrt{|x|}$

$x > 0 \quad \dot{x} = \sqrt{x}$

$x < 0 \quad \dot{x} = \sqrt{-x}$

$$\frac{d}{dx}\sqrt{|x|} = \begin{cases} x > 0 & \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} \\ x < 0 & \frac{d}{dx}\sqrt{-x} = -\frac{1}{2\sqrt{-x}} \end{cases}$$

$$\frac{d}{dx}\sqrt{|x|} = \frac{\operatorname{sign} x}{2\sqrt{|x|}}$$

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$x \geq 0$

$$\frac{dx}{dt} = \sqrt{x} \Rightarrow \int \frac{dx}{\sqrt{x}} = \int dt$$

$$2\sqrt{x} = \int \frac{dx}{\sqrt{x}} = t + c$$

$$\sqrt{x} = \frac{t+c}{2}$$

$$x(t) = \left(\frac{t+c}{2}\right)^2$$

negative $x(t) < 0$

$$x(t) = -\left(\frac{t+c}{2}\right)^2$$

$x(t) = 0$ is a solution

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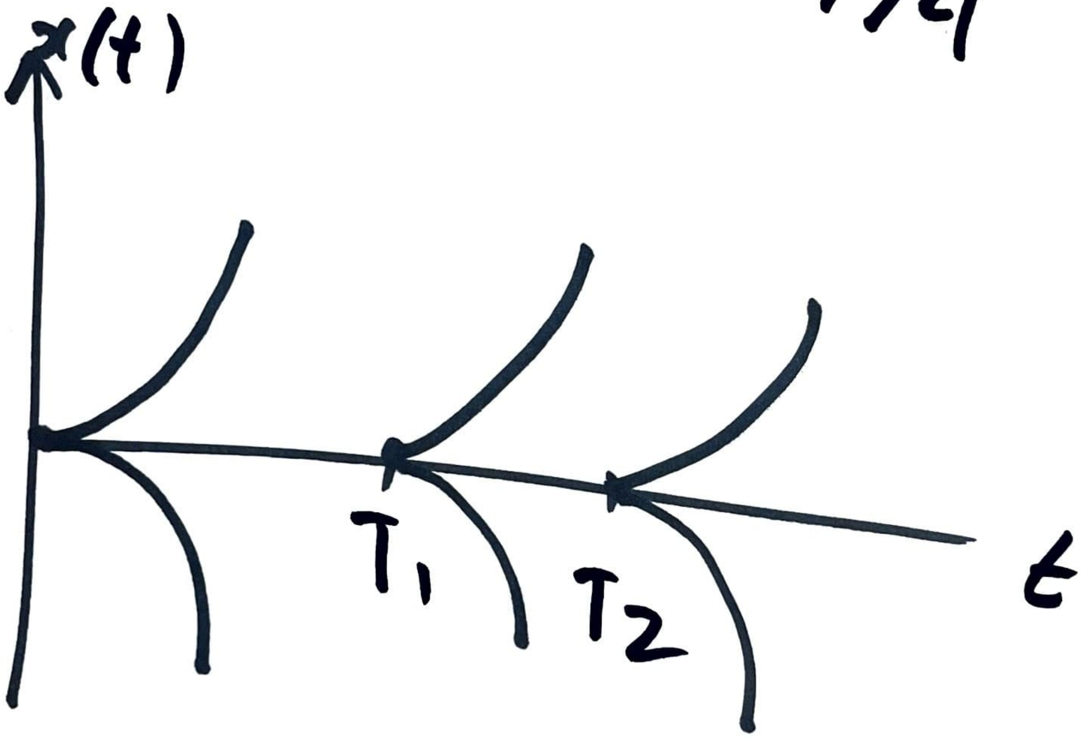
$$x(t) = \begin{cases} \emptyset, & x < T \\ \pm (x-T)^2/4 \end{cases}$$



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$$\dot{x} = 1 + x^2$$

$$\int \frac{dx}{1+x^2} = \operatorname{ArcTan} x = t + C$$

$$x = \operatorname{Tan}(t + C)$$