-1-

$\dot{x}(t) = f(x(t))$;

→ Find all fixed point
$f(x_i^*) = \emptyset$

→ consider area between fixed points

$f(x) > \emptyset$, $\dot{x} > \emptyset$, Right
$f(x) < \emptyset$ $\dot{x} < \emptyset$ Left

→ Analyze stability of fixed points.

Potential

$$\dot{x}(t) = f(x(t)) = -\frac{d}{dx} V(x)$$

$$V(x) = -\int_{0}^{x} d\tilde{x}\, f(\tilde{x})$$

$$\frac{d}{dt} x(t) = x^3(t) = -\frac{d}{dx} V(x)$$

$$V(x) = -\int_0^x y^3 \, dy = \frac{x^4}{4}$$

$$= -\frac{x^4}{4}$$

-3-

$$\dot{x}(t) = -\frac{d}{dx} V(x)$$

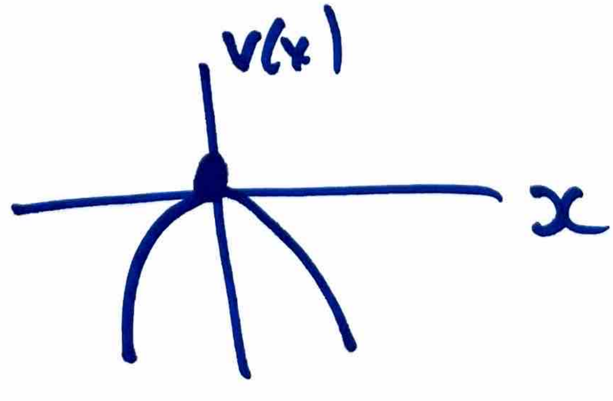
$$\frac{d}{dt} V(x(t)) = \frac{dV(x)}{dx} \cdot \dot{x}(t) =$$

$$= -\frac{dV(x)}{dx} \cdot \frac{dV(x)}{dx}$$

$$= -\left(\frac{dV(x(t))}{dx}\right)^2 \leq 0$$

Potential ALWAYS Decreases or stays the same (on a fixed point)

$$\frac{d}{dt} x(t) = \frac{d}{dx} \frac{x^4}{4}$$



Example

$\dot{x}(t) = x(t) - x^3(t)$

Find potential, fixed points, analyze stability.

Solution

$\dot{x} = f(x)$; $f(x) = x - x^3$

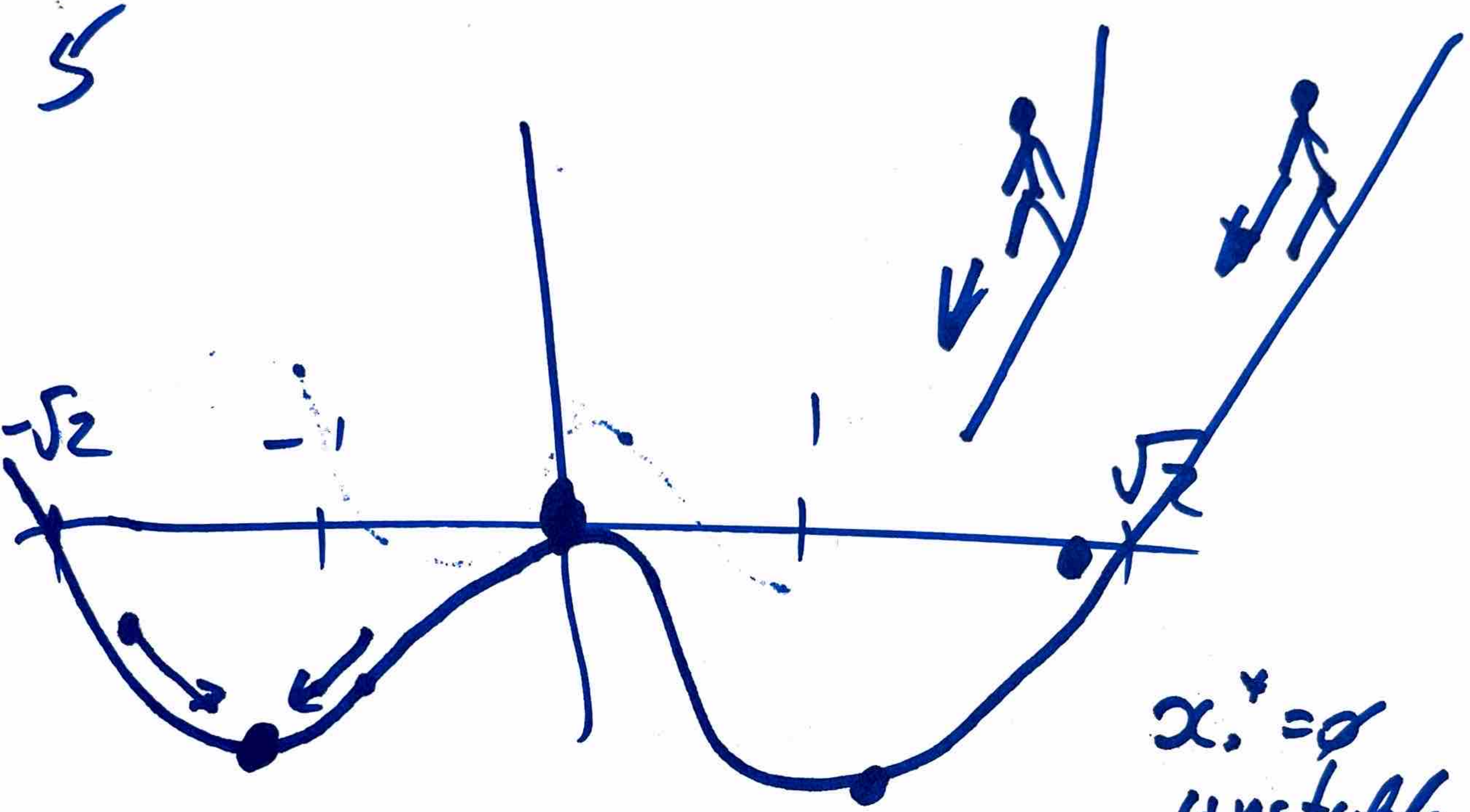
$$V(x) = -\int_0^x (\xi - \xi^3)\, d\xi$$

$$= -\frac{x^2}{2} + \frac{x^4}{4} + C; \quad C = \emptyset;$$

$$= -\frac{x^2}{2}\left(1 - \frac{x^2}{2}\right)$$

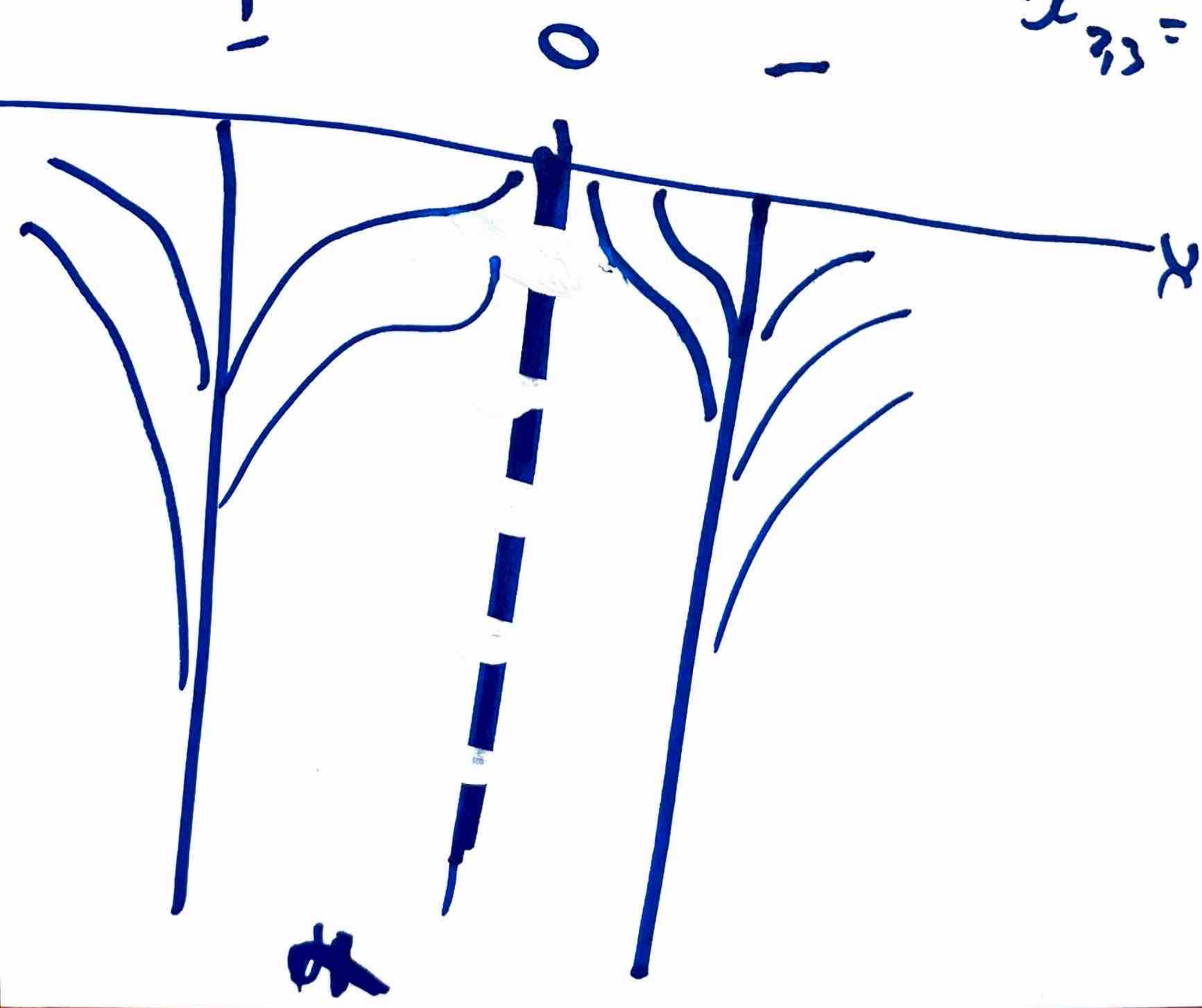
$V(x^*) = \emptyset$

$x^*_{1,2} = \emptyset$; $x^*_{3,4} = \pm\sqrt{2}$

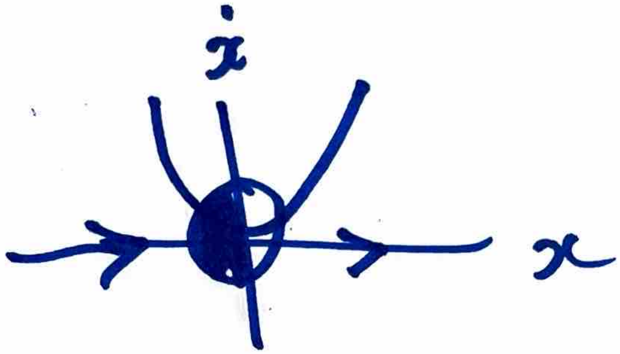


$x_0^* = 0$
unstable

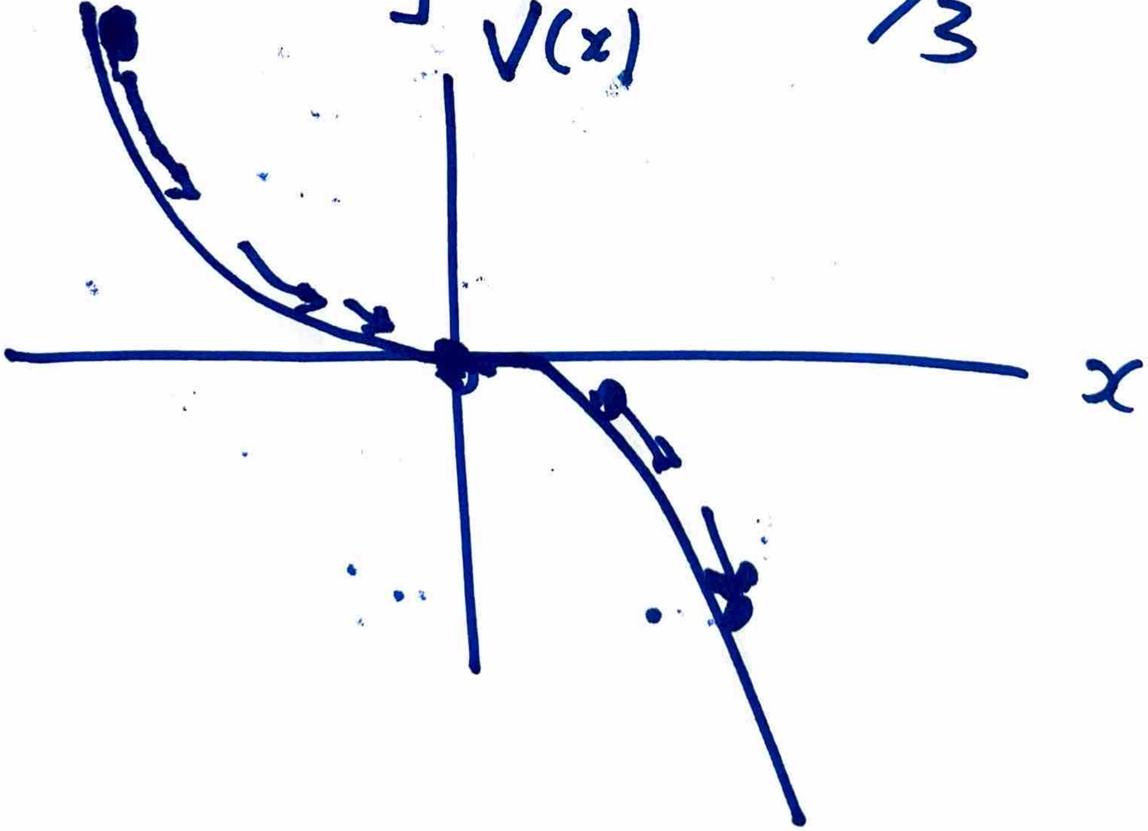
$x_{2,3}^* = \pm 1$



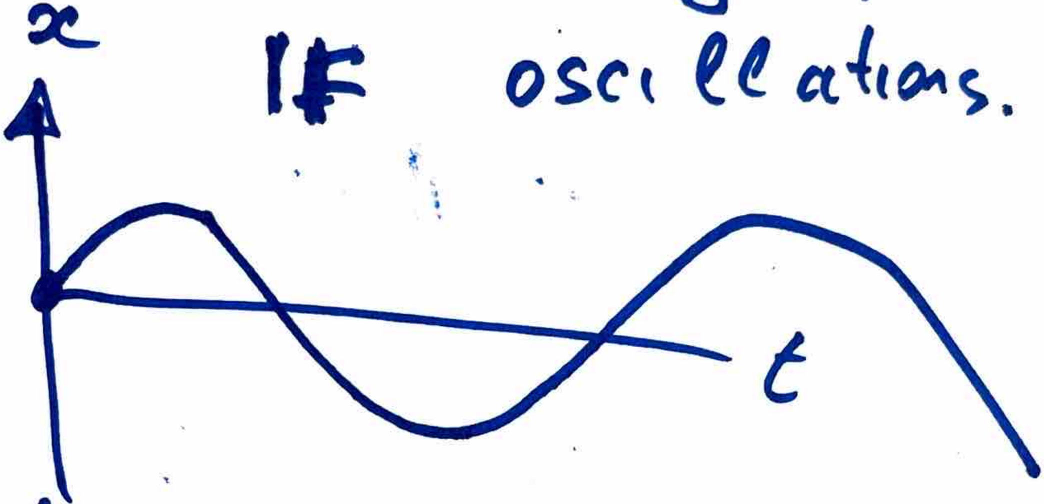
$$\dot{x} = x^2$$

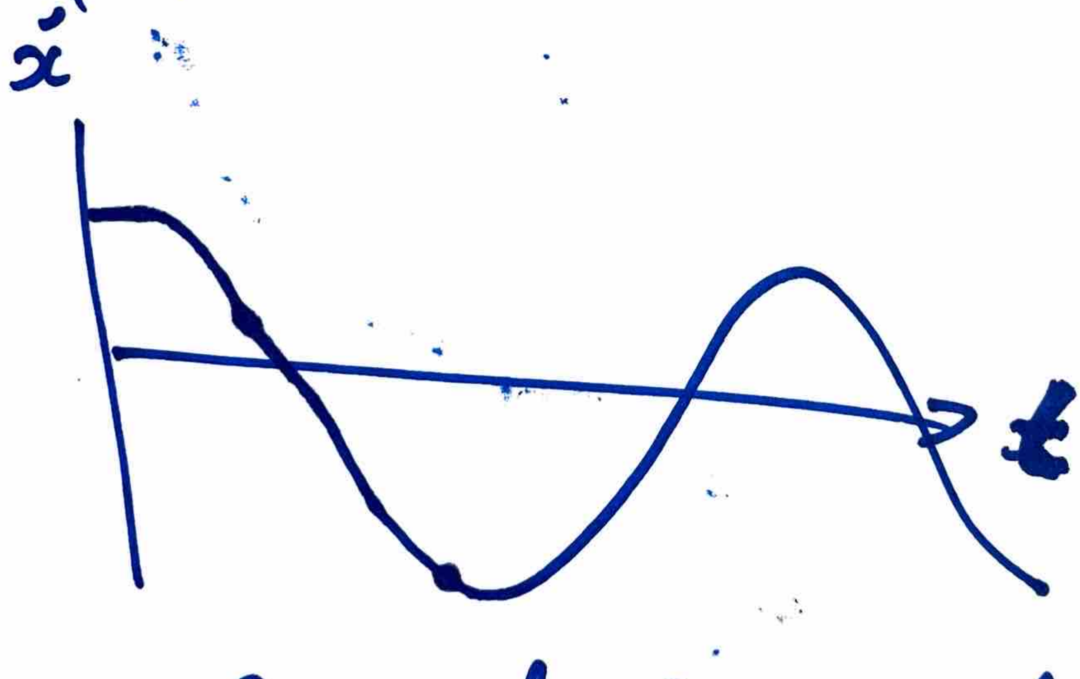


$$V(x) = -\int x^2\,dx = -\,x^3/3$$

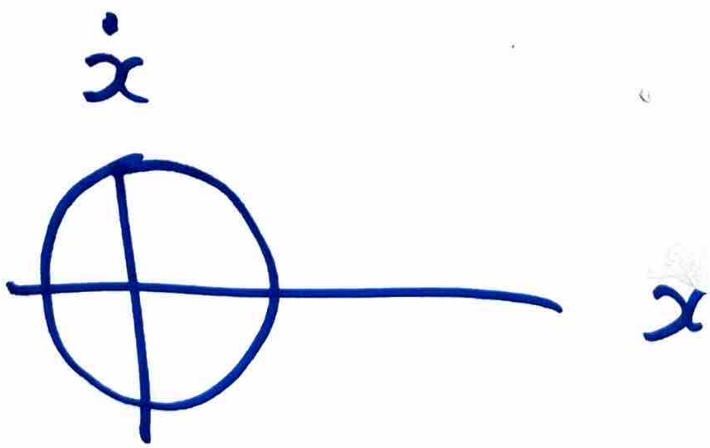


Impossibility of 1F oscillations.





Sign of $\dot{x}$ oscillates



In oscillating solutions $\dot{x}$ does change sign. Then for given x there should be two values of $\dot{x}$ - positive and negative

<u>But</u> sign of $\dot{x}$ is fixed by $f(x)$ $\Rightarrow$ contradiction

no oscillations

$$\dot{x} = -\frac{dv(x(t))}{dx}$$

Showed potential can not increase.

In oscillations potential should oscillate -> impossible

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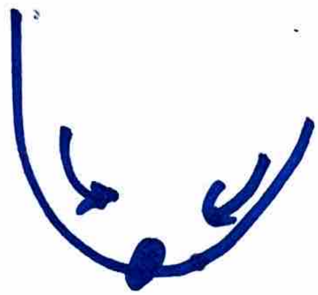
$$m\ddot{x} + b\dot{x} + kx = \emptyset,$$

overdamped

$b \gg 1, \; |m| \ll 1$



large viscosity.

Example

$V(x) = 1$

$$\dot{x} = \emptyset$$

Bifurcation

$$\dot{x}(t) = f(x(t), r)$$

external parameter

$r \equiv$ Mass

Saddle Node Bifurcation

$\dot{x}(t) = r + x^2(t)$

r is tunable parameter

tunable

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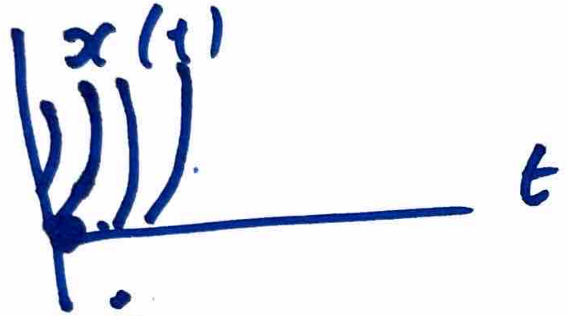
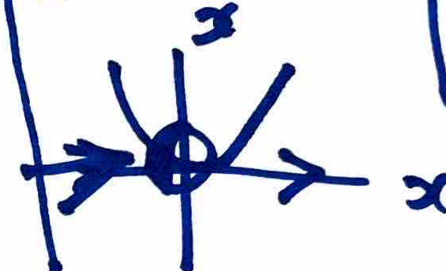
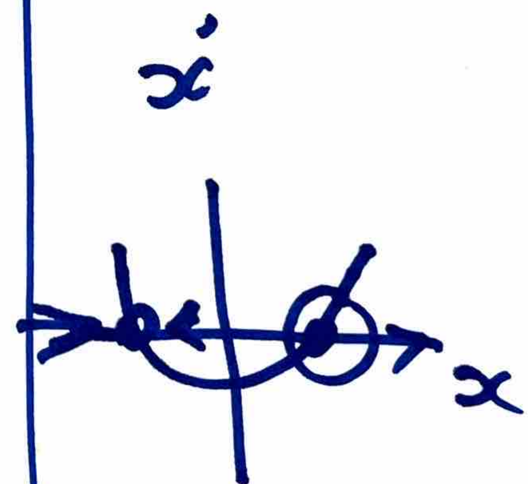
$\dot{x} = r + x^2$

$r > 0$; No fixed points
$x(t)$ is increasing



$r = 0$;
$x^* = 0$
$\dot{x} = x^2$;

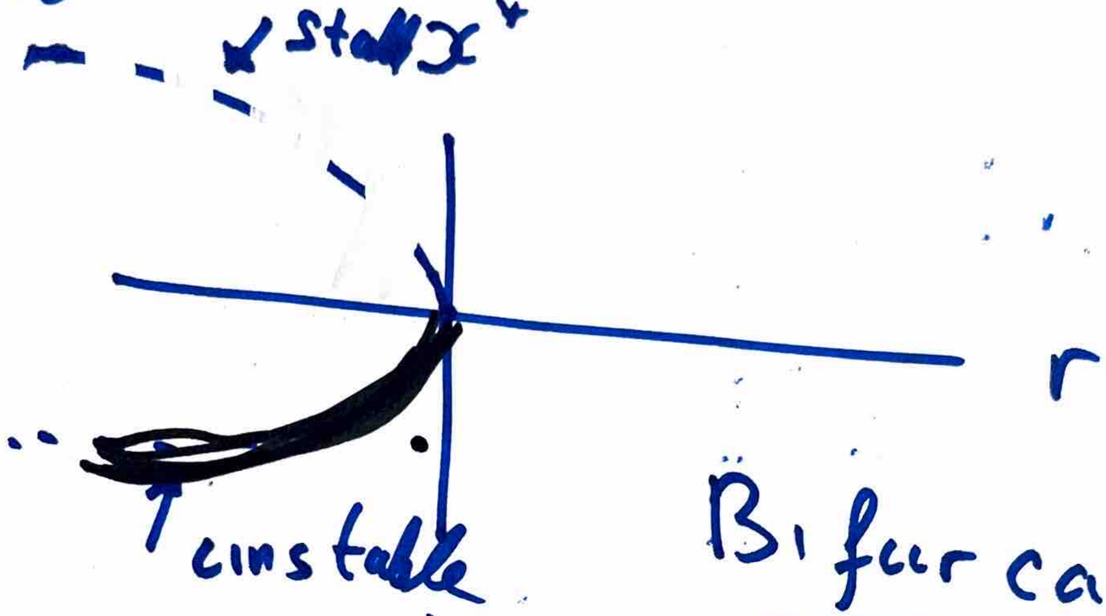




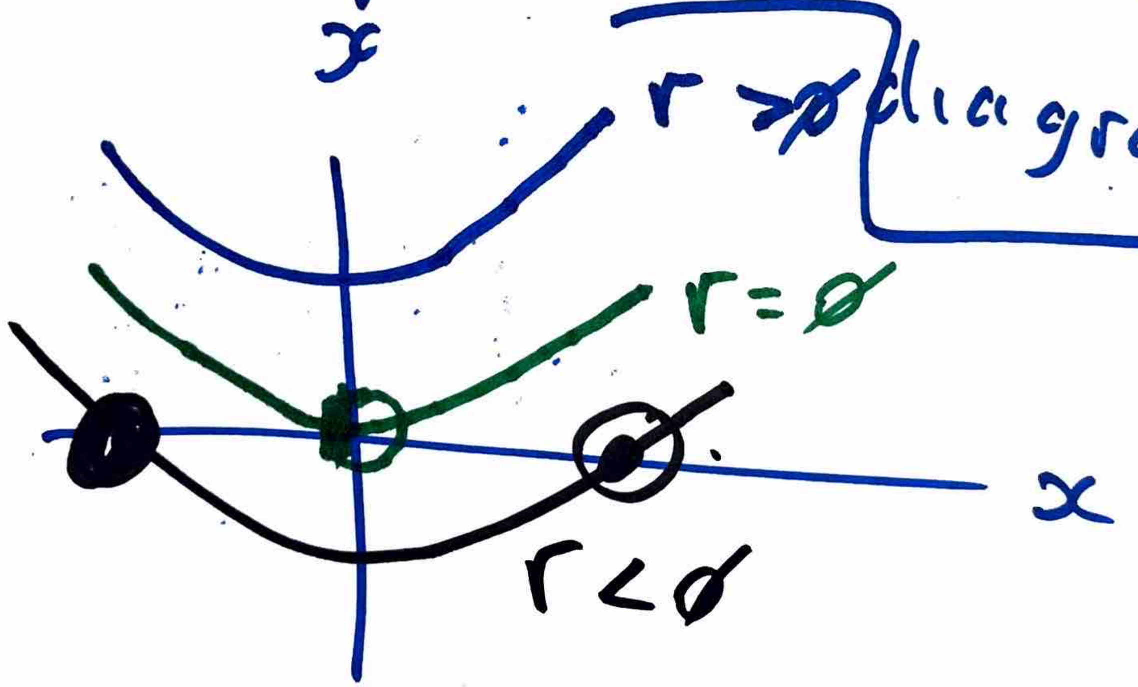
$r < 0$
$x^* = \pm\sqrt{-r}$
$f(x) = r + x^2$
$f'(x) = 2x$

$x^* = \sqrt{-r}$; $f'(x^*) > 0$
unstable

$x^* = -\sqrt{-r}$
$f'(x) < 0$
stable



Bifurcation diagram



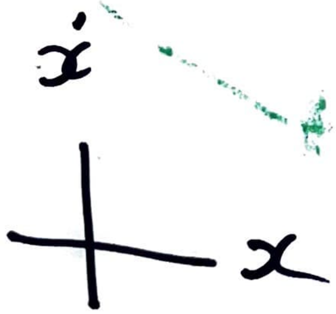
$$\dot{x} = r - x - e^{-x}$$

Analyze system, find fixed points, find and draw bifurcation diagram.

Find Fixed points

$$\emptyset = r - x - e^{-x}$$

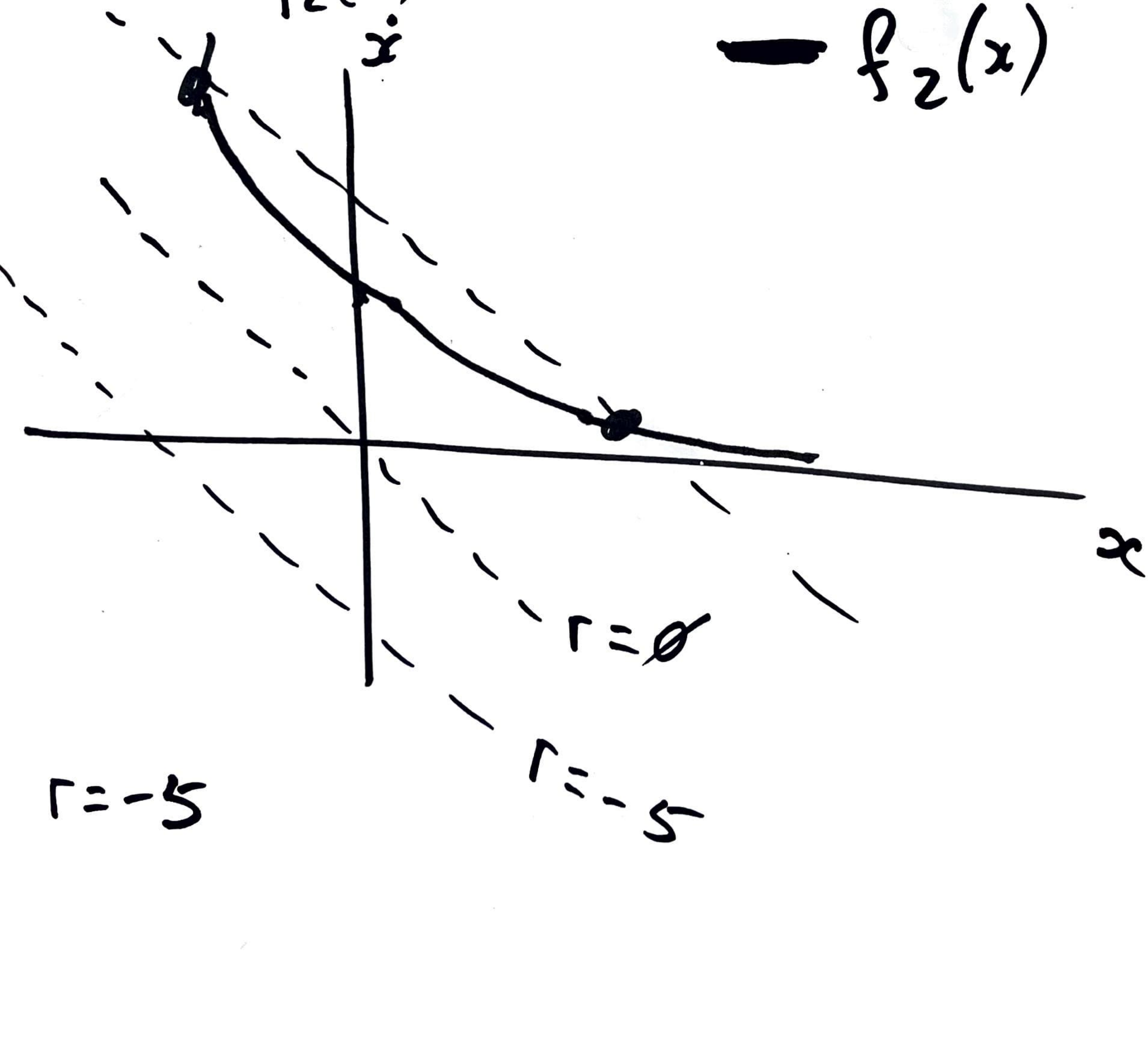
$$r = x^* + e^{-x^*} \quad ; \quad x^* = x^*(r)$$



$$\dot{x} = f_1(x, r) - f_2(x)$$

$$f_1(x, r) = r - x$$

$$f_2(x) = e^{-x}$$



18 $f_1(x,r) = r - x$

$f_2(x) = e^{-x}$

Bifurcation appears, when

$f_1(x^*, r^*) = f_2(x^*)$

and

$\frac{\partial}{\partial x} f_1(x^*, r^*) = \frac{d}{dx} f_2(x^*)$

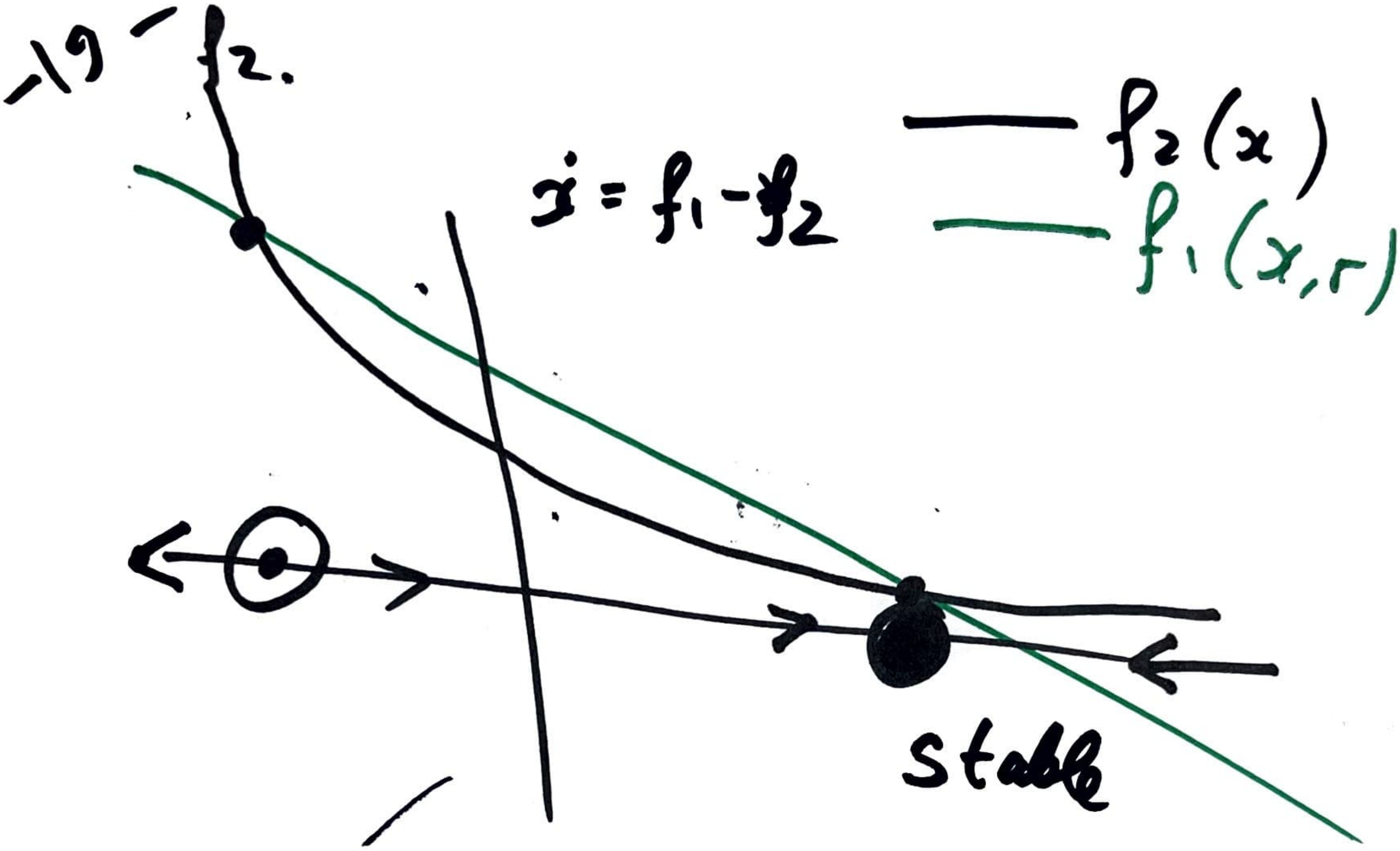
$\frac{\partial}{\partial x} f_1(x^*, r^*) = \frac{d}{dx} f_2(x^*)$

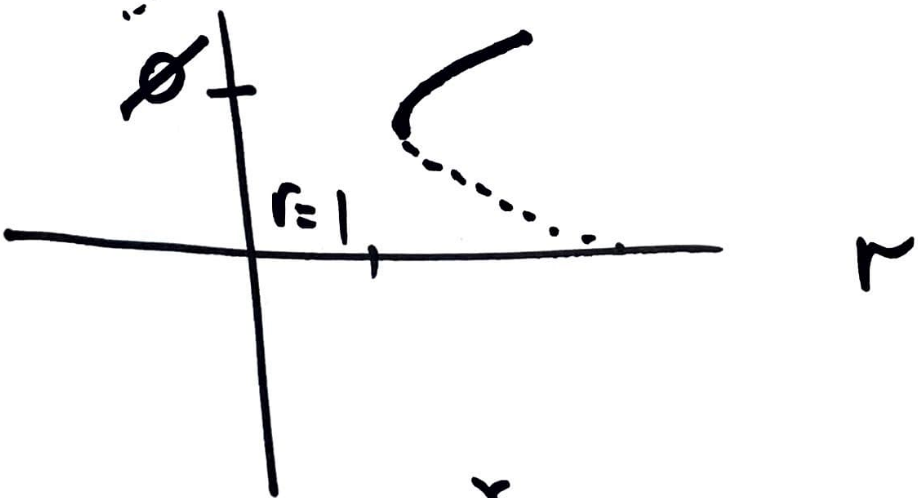
$r - x^* = e^{-x^*}$

$-1 = -e^{-x^*} \rightarrow 1 = e^{-x^*}$

$x^* = \emptyset$

$r - \emptyset = e^{\emptyset} = 1$ $(x^*, r^*) = (\emptyset, 1)$





$$r = x + e^{-x}$$