$$\frac{d}{dt}\underline{x}(t) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \underline{x}(t)$$

$$\underline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\lambda_{1,2} = \frac{Tr(A) \pm \sqrt{Tr^2(A) - 4\,Det(A)}}{2}$$

$$Tr(A) = \tau$$

$$Det(A) = \Delta$$

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

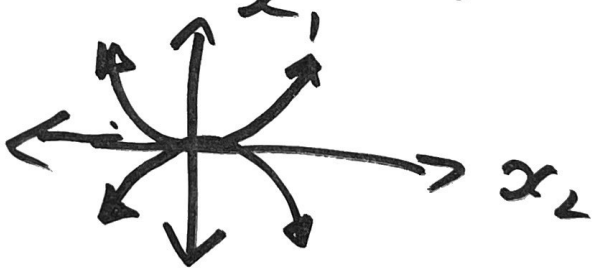
$$\tau^2 > 4\Delta$$

$$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4\Delta}}{2a} \quad ; \quad T^2 > 4\Delta$$

(I) $\frac{T^2}{4} > \Delta > 0$

(a) $T > 0$, 2 positive roots

$\lambda_1 \geq \lambda_2 > 0$



(b) $T < 0$

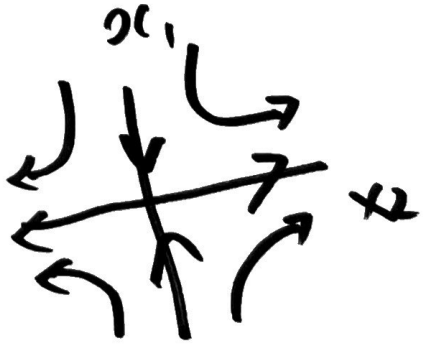
$\lambda_1 \leq \lambda_2 < 0$



(II)

$\Delta < 0$

$\lambda_1 < 0 < \lambda_2$



-3-

$$\tau^2 < 4\Delta$$

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \frac{\tau \pm i\sqrt{4\Delta - \tau^2}}{2}$$

$$\gamma = \tau/2;$$

$$\omega = \sqrt{4\Delta - \tau^2}/2$$

$$\lambda_{1,2} = \gamma \pm i\omega; \quad \gamma = Re(\lambda)$$
$$\omega = Im(\lambda)$$

4

$$e^{ix} = \cos x + i \sin x$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\sin x \cdot \cos x = \frac{1}{4i}\left(e^{ix} + e^{-ix}\right)\left(e^{ix} + e^{-ix}\right)$$

$$= \frac{1}{2 \cdot 2i}\left(e^{2ix} - e^{-2ix}\right)$$

$$= \frac{1}{2} \sin 2x$$

5

$$\lambda_{1,2} = \gamma \pm i\omega$$

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$$

$$\underline{x}(t) = \underline{x}_1 \cdot C_1 \cdot e^{\lambda_1 t} + \underline{x}_2 \cdot C_2 \cdot e^{\lambda_2 t}$$

$$= C_1\, \underline{x}_1\, e^{\gamma t + i\omega t}$$

$$+ C_2\, \underline{x}_2\, e^{\gamma t - i\omega t}$$

Let $\lambda_1 = \gamma + i\omega$

$\lambda_2 = \gamma - i\omega = \bar{\lambda}_1$

$$\underline{\underline{A}}\, \underline{x}_1 = \lambda_1\, \underline{x}_1 \leftarrow CC, \quad \underline{Im}(A) = \emptyset$$

$$\underline{\underline{A}}\, \bar{\underline{x}}_1 = \bar{\lambda}_1\, \bar{\underline{x}}_1 = \lambda_2\, \bar{\underline{x}}_1 \qquad \underline{x}_2 = \bar{\underline{x}}_1$$

$$\underline{x}_1 = \underline{a} + i\underline{b}$$

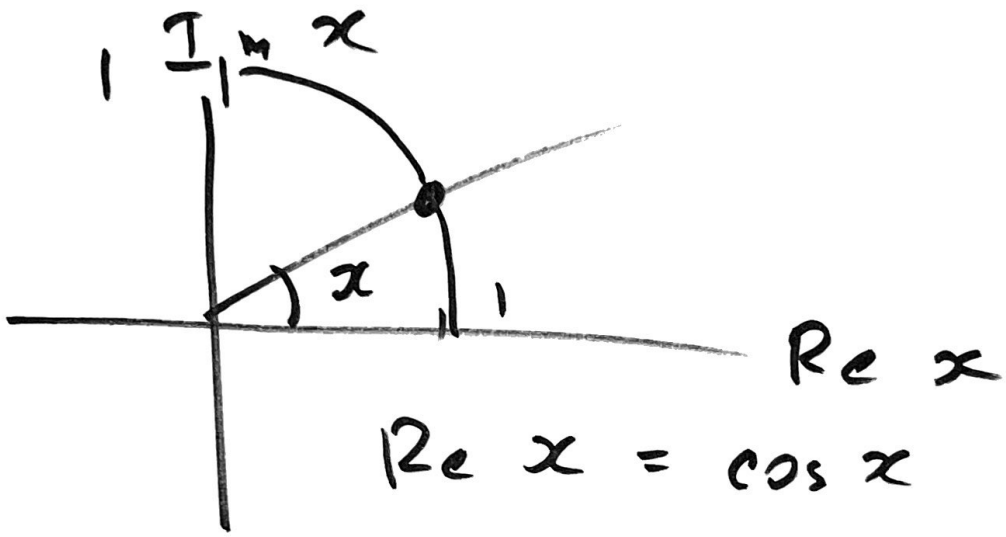
$$\underline{a} = \mathrm{Re}\ \underline{x}_1$$

$$\underline{b} = \mathrm{Im}\ \underline{x}_1$$

$\lambda_1 = \gamma + i\omega$; $\underline{x}_1 = \underline{a} + i\underline{b}$

$\lambda_2 = \gamma - i\omega$; $\underline{x}_2 = \underline{a} - i\underline{b}$

$$e^{ix} = \cos x + i \sin x$$



$$\text{Re } x = \cos x$$

$$\text{Im } x = \sin x$$

$$e^{-ix} = \cos(-x) + i \sin(-x)$$

$$= \cos x - i \sin x$$

$$\underline{x}(t) = C_1 \underline{x}_1 e^{\lambda_1 t} + C_2 \underline{x}_2 e^{\lambda_2 t}$$

$$= C_1 (\underline{a} + i\underline{b}) e^{\gamma t + \omega i t} + C_2 (\underline{a} - i\underline{b}) e^{\gamma t - i\omega t}$$

$$= C_1 (\underline{a} + i\underline{b}) e^{\gamma t} (\cos\omega t + i \sin\omega t) + C_2 (\underline{a} - i\underline{b}) e^{\gamma t} (\cos\omega t - i \sin\omega t)$$

$$= C_1 e^{\gamma t} (\underline{a}\cos\omega t + i\underline{a}\sin\omega t + i\underline{b}\cos\omega t - \underline{b}\sin\omega t) + C_2 e^{\gamma t} (\underline{a}\cos\omega t - \underline{a}i\sin\omega t - i\underline{b}\cos\omega t - \underline{b}\sin\omega t)$$

$$= e^{\gamma t} [(\underline{a}\cos\omega t)(C_1 + C_2) + i\underline{b}\cos\omega t (C_1 - C_2) + i\underline{a}\sin\omega t (C_1 - C_2) - \underline{b}\sin\omega t (C_1 + C_2)]$$

$$C_1 + C_2 = D_1$$

$$i(C_1 - C_2) = D_2$$

$$\underline{x}(t) = e^{\gamma t} [\underline{a} D_1 \cos \omega t + \underline{\beta} D_2 \cos \omega t + \underline{a} D_2 \sin \omega t - \underline{\beta} D_1 \sin \omega t)$$

$$x(t) = e^{\gamma t} \cdot \cos \omega t [\underline{a} D_1 + \underline{\beta} D_2] + e^{\gamma t} \sin \omega t [a D_2 - \underline{\beta} D_1]$$

$$= e^{\gamma t} D_1 (\underline{a} \cos \omega t - \underline{\beta} \sin \omega t] + e^{\gamma t} D_2 (\underline{a} \sin \omega t + \underline{\beta} \cos \omega t)$$

$$\lambda_1 = \gamma + i\omega$$

$$\lambda_2 = \gamma - i\omega$$

$$\underline{x}_1 = \underline{a} + i\underline{b}$$

$$\underline{a} = \mathrm{Re}\, \underline{x}_1 \qquad \underline{b} = \mathrm{Im}\, \underline{x}_1$$

$$\mathrm{Im}\, \underline{\underline{A}} = 0$$

$$\underline{\underline{A}} \cdot \underline{x}_1 = \lambda_1 \cdot \underline{x}_1$$

$$\underline{\underline{A}}^* \, \underline{x}_1^* = \lambda_1^* \, \underline{x}_1^* \qquad \alpha_1^* \equiv \text{C.C.}$$

If $\mathrm{Im}(A) = 0$, $A^* = A$

$$\underline{\underline{A}} \, \underline{x}_1^* = \lambda_1^* \, \underline{x}_1^* \qquad \underline{x}_1^* = \underline{x}_2$$

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$$a y'' + b y' + c y = 0$$

$$y \equiv y(x)$$

$$y(x) = e^{\lambda x}$$

$$a\lambda^2 + b\lambda + c = 0$$

~~$\lambda_{1,2} = i\omega \pm$~~

$$\lambda_{1,2} = \gamma \pm i\omega$$

$$y(x) = C_1 e^{\gamma x} \cos \omega x + C_2 e^{\gamma x} \sin \omega x$$

12

$$A = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\begin{pmatrix} x(t=0) \\ y(t=0) \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$$0 = Det(A - \lambda I) =$$

$$= Det\begin{bmatrix} 3-\lambda & 9 \\ -4 & -3-\lambda \end{bmatrix}$$

$$= (3-\lambda)(-3-\lambda) + 36 = 0$$

$$(\lambda - 3)(\lambda + 3) + 36 = 0$$

$$\lambda^2 - 9 + 36 = 0$$

$$\lambda^2 = -27 = -9 \cdot 3$$

$$\lambda_{1,2} = \pm i\, 3\sqrt{3}$$

13 $\underline{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\underline{\underline{A}}\underline{x} = \lambda \underline{x}$

$$\begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 3\sqrt{3}\,i \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$3\alpha + 9\beta = 3\sqrt{3}\,i \cdot \alpha$

$\alpha + 3\beta = \sqrt{3}\,i\,\alpha$

$\alpha(1 - \sqrt{3}\,i) = -3\beta$

$\alpha = -3$

$\beta = 1 - \sqrt{3}\cdot i$ $\beta = \frac{1-\sqrt{3}\,i}{-3}\alpha$

$$\underline{x} = \begin{pmatrix} -3 \\ 1-\sqrt{3}\cdot i \end{pmatrix}$$

$\lambda_1 = \lambda_2^* = 3\sqrt{3}\,i$

$\underline{x} = \underline{a} + i\underline{b}$, $\underline{a} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}$

14

$$\underline{x}(t) = D_1 e^{\gamma t}\left[a\cos\omega t - b\sin\omega t\right] + D_2 e^{\gamma t}\left[a\sin\omega t + b\cos\omega t\right]$$

$\omega = 3\sqrt{3}$

$\gamma = 0$

$$a = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}$$

$$\underline{x}(t) = D_1\left[\begin{pmatrix} -3 \\ 1 \end{pmatrix}\cos 3\sqrt{3}t - \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}\sin 3\sqrt{3}t\right] + D_2\left[\begin{pmatrix} -3 \\ 1 \end{pmatrix}\sin 3\sqrt{3}t + \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}\cos 3\sqrt{3}t\right]$$

$$x(t=0) = D_1\begin{pmatrix} -3 \\ 1 \end{pmatrix} + D_2\begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$D_1 = -2/3$

$-D_2\cdot\sqrt{3} = -4 - D_1 = -4 + 2/3$

$D_2 = 4/\sqrt{3} - \frac{2}{3\sqrt{3}}$

$$\frac{d}{dt}\underline{x} = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix} \underline{x}(t)$$

$$\underline{x}(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\frac{d}{dt}x(t=0) = \begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

