$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$x(t) = D_1 e^{\gamma t}(a\cos\omega t - b\sin\omega t) + D_2 e^{\gamma t}(b\cos\omega t + a\sin\omega t)$$

$\gamma = \mathrm{Re}\,\lambda$; $\omega = \mathrm{Im}(\lambda)$;

-2-

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\begin{pmatrix} x(t=0) \\ y(t=0) \end{pmatrix} = \begin{pmatrix} 3 \\ -10 \end{pmatrix}$$

$$\text{Det}\,[A - \lambda I] = 0$$

$$0 = \text{Det}\begin{bmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{bmatrix} =$$

$$= (\lambda-3)(\lambda-1) + 65$$

$$= \lambda^2 - 4\lambda + 68 = 0$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 4\cdot 68}}{2} = \frac{4 \pm \sqrt{-256}}{2}$$

$$= 2 \pm 8i$$

-3-

$$\begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (2+8i)\begin{pmatrix} a \\ b \end{pmatrix}$$

$$3a - 13b = (2+8i)a$$
$$-13b = (-1+8i)a$$

$$a = -13$$

$$b = -1+8i \qquad \underline{x}_1 = \begin{pmatrix} -13 \\ -1+8i \end{pmatrix}$$

$$\underline{a} = Re\, x_1 = \begin{pmatrix} -13 \\ -1 \end{pmatrix} \qquad \gamma = 2$$

$$\underline{b} = Im\, x_1 = \begin{pmatrix} 0 \\ 8 \end{pmatrix} \qquad \omega = 8$$

$$\underline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$= D_1 e^{2t}\left(\begin{pmatrix} -13 \\ -1 \end{pmatrix}\cos 8t - \begin{pmatrix} 0 \\ 8 \end{pmatrix}\sin 8t\right)$$
$$+ D_2 e^{2t}\left(\begin{pmatrix} 0 \\ 8 \end{pmatrix}\cos 8t + \begin{pmatrix} -13 \\ -1 \end{pmatrix}\sin 8t\right)$$

-4-

$$\begin{pmatrix} x(t=0) \\ y(t=0) \end{pmatrix} = D_1 \begin{pmatrix} -13 \\ -1 \end{pmatrix} + D_2 \begin{pmatrix} 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ -10 \end{pmatrix}$$

$D_1 = -3/13$

$$D_2 = \frac{D_1 - 10}{8} = \frac{-3/13 - 10}{8}$$

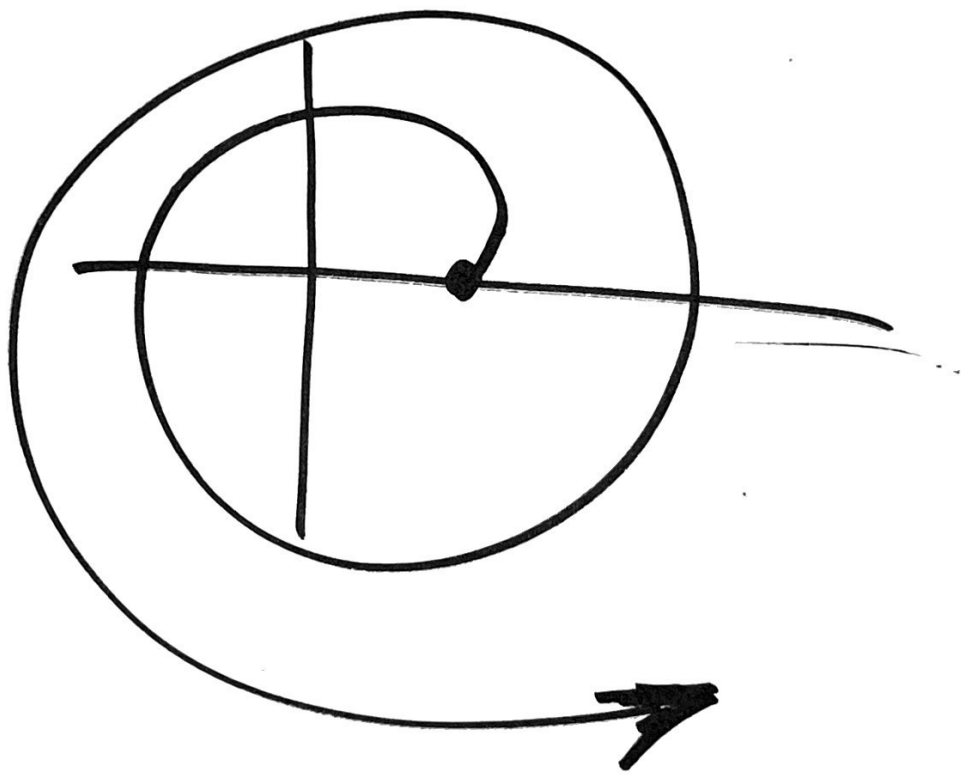
$$= \frac{-133}{13 \cdot 8}$$

$$-D_1 + 8D_2 = -10$$

$$8D_2 = -10 + D_1 = -10 - \frac{3}{13} = -\frac{133}{13}$$

$$D_2 = -\frac{133}{13 \cdot 8}$$

$$\begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$



-6-

$$A = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\text{Det}(A - \lambda I)$$

$$= \text{Det}\left[\begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= \text{Det}\begin{bmatrix} 1-\lambda & -1 \\ 4 & -3-\lambda \end{bmatrix} =$$

$$= (1-\lambda)(-3-\lambda) + 4$$

$$= (\lambda-1)(\lambda+3) + 4 = \lambda^2 + 2\lambda + 1 =$$

$$= (\lambda+1)^2 = \emptyset \; ; \; \boxed{\lambda = -1}$$

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$$\begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} = -1\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} -r \\ -s \end{pmatrix}$$

$r - s = -r$ $\quad 2r = s$

$4r - 3s = -s$ $\quad 4r = 2s$

$s = 2$

$r = 1$

$$\underline{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$\underline{x}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{-t} + c_2 \cdot ?$$

8

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$$

$$\underline{\underline{A}}\,\underline{v} = \lambda\,\underline{v}$$

$$x(t) = C_1\,\underline{v}\,e^{\lambda t}$$

$$\underline{x}(t) = \underline{a}\cdot t\,e^{\lambda t} + \underline{b}\,e^{\lambda t}$$

$$\dot{\underline{x}}(t) = \underline{a}\,e^{\lambda t} + \underline{a}\,t\lambda\,e^{\lambda t} + \underline{b}\lambda\,e^{\lambda t}$$

$$\left(\underline{a} + \underline{a}\,t\lambda + b\lambda\right)e^{\lambda t} = \underline{\underline{A}}\left(\underline{a}t + \underline{b}\right)e^{\lambda t}$$

$$\underline{a} + \underline{a}\,t\lambda + b\lambda = \underline{\underline{A}}\,\underline{a}t + \underline{\underline{A}}\,\underline{b}$$

$\underline{a} = \underline{x}$, eigenvector

$$\underline{x} + b\lambda = \underline{\underline{A}}\,\underline{b}$$

$$\underline{\underline{A}}\cdot\underline{b} - \underline{b}\lambda = \underline{x} \Rightarrow \boxed{\left(\underline{\underline{A}} - \lambda\underline{\underline{I}}\right)\underline{b} = \underline{x}}$$

9 $A = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}$; $\lambda = -1$; $\underline{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{b} = \underline{x}$

$A - \lambda I = (\underline{\underline{A}} + \underline{\underline{I}}) = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$

$\underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$; $(A - \lambda I)b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$2b_1 - b_2 = 1$

$4b_1 - 2b_2 = 2$

$b_2 = 2b_1 - 1$

$b_1 = 1, \; b_2 = 1$

$\underline{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

10

$$x(t) = c_1 \underline{x} e^{\lambda t} + c_2 (\underline{x} \cdot t + \underline{b}) e^{\lambda t}$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{\lambda t} + c_2 \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) e^{\lambda t}$$

11

$$(x-1)^2 = x^2 - 2x + 1$$

$$y''(x) - 2y'(x) + y(x) = 0$$

$$y(x) = e^{\lambda t}$$

$$(\lambda^2 - 2\lambda + 1) = 0$$

$$\lambda = 1,$$

$$y(x) = C_1 \cdot e^t + C_2 \cdot t e^t$$

11

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\text{Det}\,(A - s\,I) =$$

$$= \text{Det}\begin{pmatrix} \lambda - s & 1 \\ 0 & \lambda - s \end{pmatrix} = (\lambda - s)^2 = 0$$

$$s = \lambda$$

Eigenvalue is λ

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\lambda x_1 + x_2 = \lambda x_1 \Rightarrow x_2 = 0$$

$$\underline{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$x(t) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t} ;$$

12

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{b} = \underline{\underline{x}}$$

$$\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\boxed{b_2 = 1}$ $\quad \underline{b} = \begin{pmatrix} b \\ 1 \end{pmatrix}$

$$x(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t} + c_2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} b \\ 1 \end{pmatrix} \right) e^{\lambda t}$$

$$= e^{\lambda t} \begin{bmatrix} c_1 + c_2 t + b \\ c_2 \end{bmatrix} \quad \text{Now } b = 0$$

-13-

a is a parameter

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\lambda_1 = -2$; $\lambda_2 = a$

$$\begin{pmatrix} -2 & 1 \\ 0 & a \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = -2\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$$

$$-2x_{11} + x_{12} = -2x_{11}$$

$x_{12} = 0$; $\underline{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

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$$\begin{pmatrix} -2 & 1 \\ 0 & a \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = a \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$$-2\, x_{21} + x_{22} = a\, x_{21}$$

$$(a+2)\, x_{21} = x_{22}$$

$$\underline{x}_2 = \begin{pmatrix} 1 \\ a+2 \end{pmatrix}$$

$$\lambda_1 = -2; \quad \underline{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = a; \quad \underline{x}_2 = \begin{pmatrix} 1 \\ a+2 \end{pmatrix}$$

~~Animate[StreamPlot[{-2x+y, ay}, {x, y, -2, 2}]~~

$$A = \begin{pmatrix} -2 & 1 \\ \emptyset & a \end{pmatrix}; \quad \lambda_1 = -2; \quad \lambda_2 = a$$

$\lambda = -2$: $\begin{pmatrix} -2 & 1 \\ \emptyset & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$ $-2x + y = -2x$

$ay = -2y$

$y = \emptyset, x = 1$ or

$\lambda = a$: $\begin{pmatrix} -2 & 1 \\ \emptyset & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = a\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow$ $(a+2)x = y$

$ay = ay \Rightarrow y = 1$

$\underline{x} = \begin{pmatrix} \frac{1}{a+2} \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ a+2 \end{pmatrix}$

Animate[StreamPlot[{-2x+y, ay}, {x, -4, 4}, {y, -4, 4}],
PlotLabel → {a}], {a, -3, 3}]

Animate[Show[StreamPlot[{-2x+y, ay},
{x, -3, 3}, {y, -3, 3}, PlotLabel → {a}],
Plot[{x/100, (a+2)x}, {x, -3, 3}],
{a, -6, 6}];

$$y(x) = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1 \\ a+2 \end{pmatrix} \cdot c_2 \cdot e^{at}.$$