18

$$r = \sqrt{x^2+y^2} = \sqrt{(x(t))^2 + (y(t))^2}$$

$$\frac{dr}{dt} = \frac{1}{2\sqrt{x^2+y^2}} \cdot \dots$$

$$\frac{d}{dt} r(t) = \frac{d}{dt}\sqrt{x^2+y^2} =$$

$$= \left(\frac{\partial}{\partial x}\sqrt{x^2+y^2}\right)\cdot \dot{x} + \left(\frac{\partial}{\partial y}\sqrt{x^2+y^2}\right)\cdot \dot{y}$$

$$= \frac{2x(-y + \alpha x(x^2+y^2))}{2\sqrt{x^2+y^2}}$$

$$+ \frac{2y(x + \alpha y(x^2+y^2))}{2\sqrt{x^2+y^2}}$$

$$= \frac{1}{r}(-xy + \alpha x^2(x^2+y^2) + xy + \alpha y^2($$

$$\dot{x} = -y + \alpha x (x^2+y^2)$$

$$\dot{y} = x + \alpha y (x^2+y^2)$$

$$\frac{d}{dt} r = \alpha r^3$$

$$\frac{d}{dt} \varphi = 1$$

Linearized $(0,0)$ $\Rightarrow$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

$\theta = \text{Arc Tan } y/x$

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f(x(t),y(t))}{\partial x}\frac{dx}{dt} + \frac{\partial f(x(t),y(t))}{\partial y}\frac{dy}{dt}$$

-2-

$$\frac{dr}{dt} = \frac{\partial r}{\partial x}\frac{dx}{dt} + \frac{\partial r}{\partial y}\frac{dy}{dt}$$

$$= \frac{x}{\sqrt{x^2+y^2}} \cdot \left(-y + \alpha x (x^2+y^2)\right)$$

$$+ \frac{y}{\sqrt{x^2+y^2}} \left(x + \alpha y (x^2+y^2)\right)$$

$$r = \sqrt{x^2+y^2} \qquad r^2 = x^2+y^2$$

$$\frac{dr}{dt} = \frac{x}{r}(-y+\alpha x r^2) + \frac{y}{r}(x + \alpha y r^2)$$

$$= \frac{1}{r}(-xy + \alpha x^2 r^2 + yx + \alpha y^2 r^2)$$

$$= \frac{1}{r}\left(\alpha r^2 (x^2+y^2)\right) = \alpha r^3$$

$$\frac{dr}{dt} = \alpha r^3$$

$$\varphi = \operatorname{Arc\,Tan}\left(y/x\right)$$

$$\frac{d}{dt}\varphi(t) = \frac{d}{dt}\operatorname{Arc\,Tan}\left(y(t)/x(t)\right)$$

$$\frac{d}{dx}\left(\operatorname{Arc\,Tan} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dt}\varphi(t) = \left(\frac{\partial}{\partial x}\operatorname{Arc\,tan} y/x\right)\dot{x} +$$

$$+ \frac{\partial}{\partial y}\operatorname{Arc\,Tan} y/x)\dot{y} =$$

$$= \frac{-y}{1+(y/x)^2}\,\frac{1}{x^2}\left(-y\dot{x} + x\dot{y}\right)$$

$$+ \frac{1}{x\left(1+(y/x)^2\right)}\left(x\dot{y} - y\dot{x}\right)$$

$$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$$

$$\frac{d}{dt}\,\psi(t) = \frac{(+y^2 - 2xy(x^2+y^2))}{x^2+y^2}$$

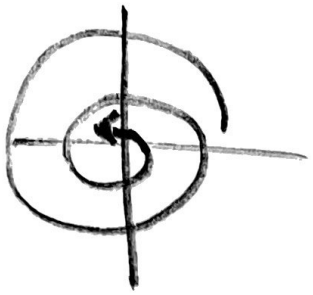
$$+\ \frac{x^2 + 2xy(x^2+y^2)}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1$$

6-1

$\alpha > 0$



$\alpha < 0$



$x(t)$ - rabbits

$y(t)$ - sheep

Lotka - Volterra system

$$\frac{d}{dt}x(t) = x(t)\cdot(3 - x(t) - 2y(t))$$

$$\frac{d}{dt}y(t) = y(t)\cdot(2 - y(t) - x(t))$$

Logistic growth

Simple population growth

$\dot{x} = x(3 - x - 2y) = f(x, y)$

$\dot{y} = y(2 - y - x)$

- $(0, 0)$ – everybody died.
- $x = 0$, $\dot{x} = 0$ $\quad 2 - y = 0 \Rightarrow y = 2$

  $(0, 2)$ – sheep

  Kingdom
- $y = 0$; $\quad x = 3$; $\quad 3 - x - 2y|_{y=0} = 0$

  $(3, 0)$

  Rabbit

  Kingdom
- $x + 2y = 3$

  $x + y = 2 \Rightarrow (x^*, y^*) = (1, 1)$

$(0, 0)$ $(0, 2)$ $(3, 0)$ $(1, 1)$

8 • when $\dot{x} = 0 \Rightarrow$

$x = 0$ or $x = 3 - 2y$

purely vertical

• when $\dot{y} = 0 \Rightarrow$

$y = 0$ or $y = 2 - x$

purely horizontal

$$J(x,y) = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{bmatrix}$$

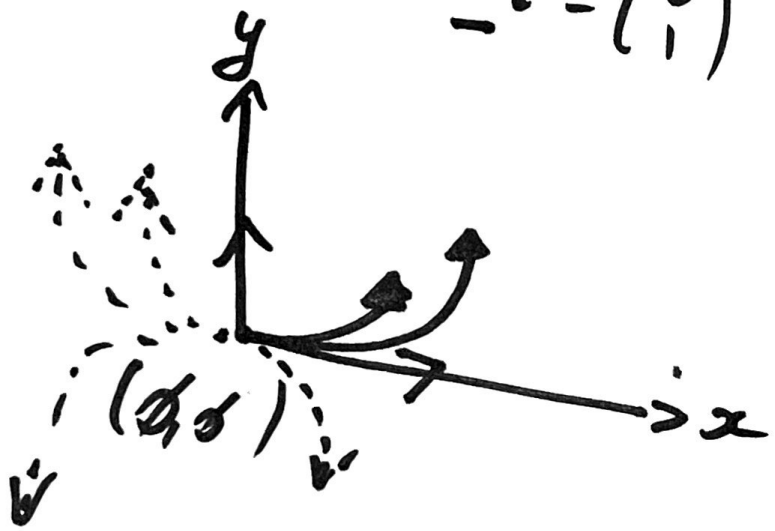
$$= \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - 2y - x \end{pmatrix}$$

-9- $x^* = y^* = 0$

$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

$\lambda_1 = 3$; $\underline{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 2$ $\underline{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



10

• $(3, 0)$

$$J(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$$

$\lambda_1 = -3;$

$$\begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = -3\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$$

$x_{12} = 0 \Rightarrow \underline{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

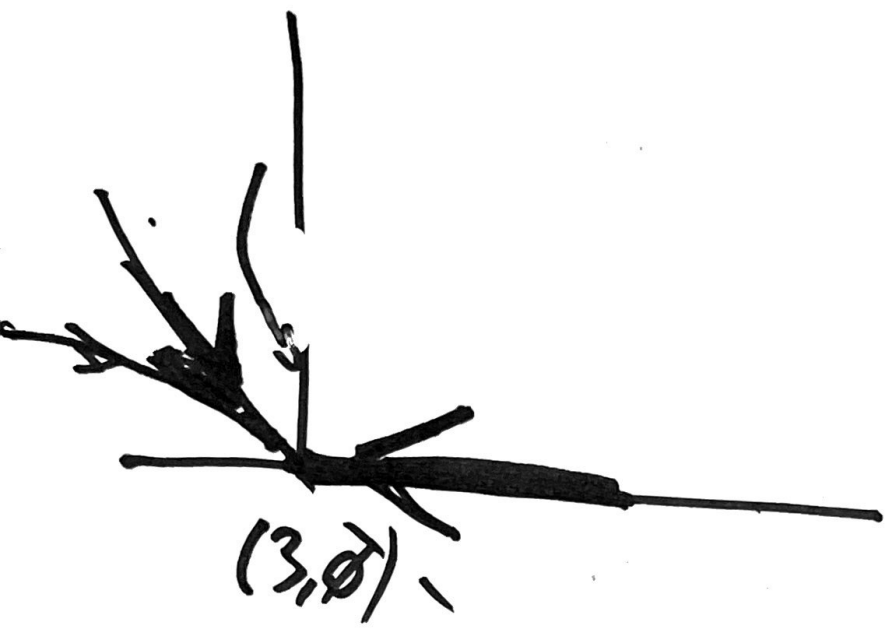
$\lambda_2 = -1;$

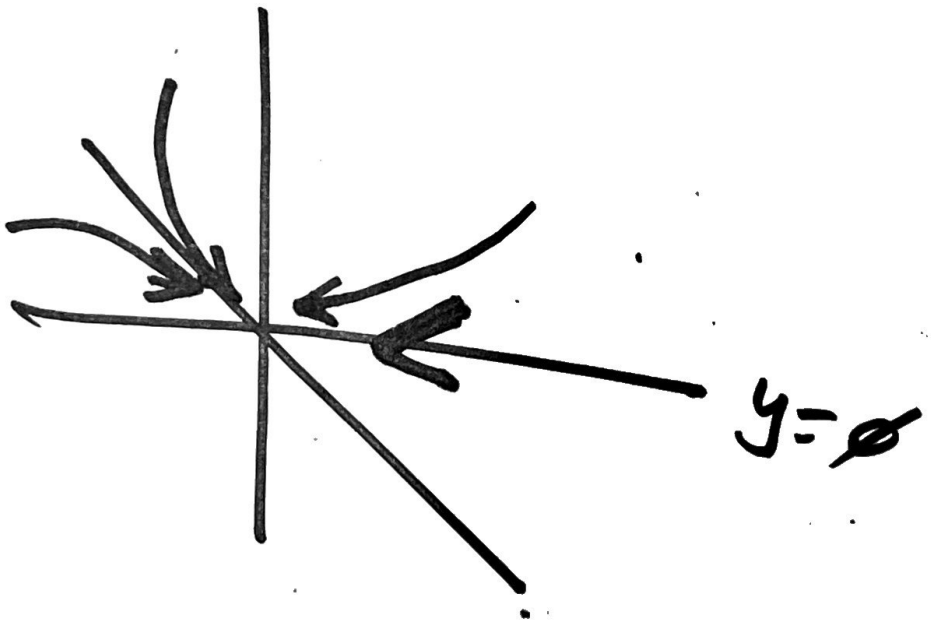
$$\begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = -\begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$-3x_{21} - 6x_{22} = -x_{21}$

$-2x_{21} = 6x_{22}$ $\quad \underline{x}_2 = \quad = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

11 (3, ∅)





42-

$$\begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} = -\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

-13-

• (1,1)

$$J(1,1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

$$(-1-\lambda)(-1-\lambda) - 2 = 0$$

$$(\lambda+1)(\lambda+1) - 2 = 0$$

$$\lambda^2 + 2\lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2}$$

$$= -1 \pm \sqrt{2}$$

$$\begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_{31} \\ x_{32} \end{pmatrix} = (-1-\sqrt{2}) \begin{pmatrix} x_{31} \\ x_{32} \end{pmatrix}$$

$$-x_{31} - 2x_{32} = -x_{31} - \sqrt{2}\, x_{31}$$

$$+2x_{32} = +\sqrt{2}\, x_{31} \Rightarrow \sqrt{2}\, x_{32} = x_{31}$$

$$\underline{x}_3 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

- $$\begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_{31} \\ x_{32} \end{pmatrix} = (-1+\sqrt{2}) \begin{pmatrix} x_{31} \\ x_{32} \end{pmatrix}$$

$$-x_{31} - 2x_{32} = -x_{31} + \sqrt{2}\, x_{32}$$

$$-2x_{32} = \sqrt{2}\, x_{31}$$

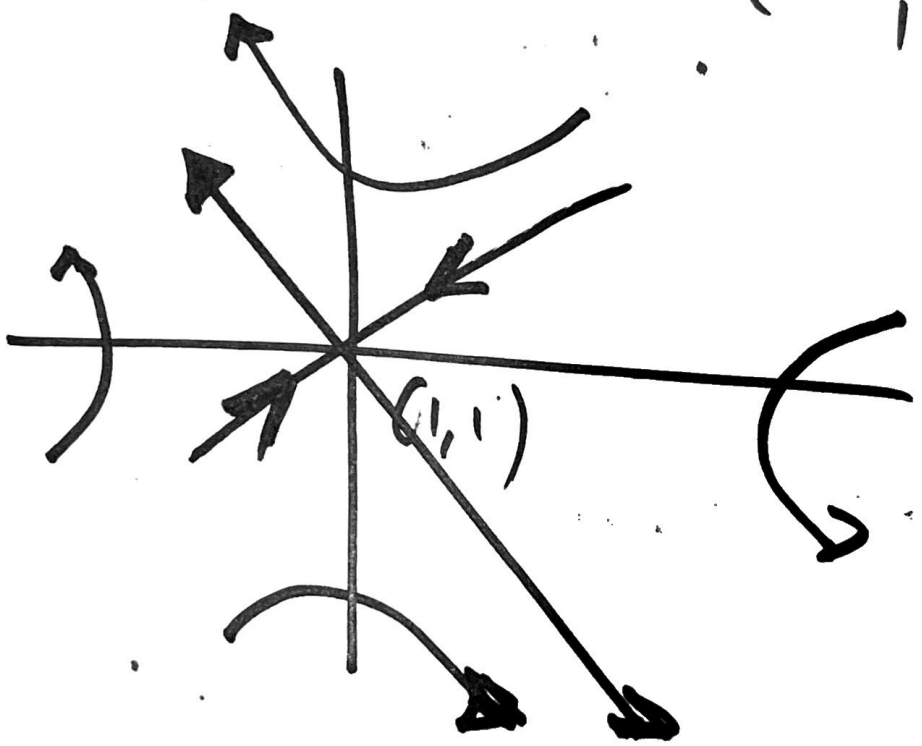
$$\underline{x}_3 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$$

• -15 -

(1, 1)

$\lambda_1 = -1 - \sqrt{2}$; $\underline{x}_1 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$

$\lambda_2 = -1 + \sqrt{2}$ $\underline{x}_2 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$



$(\emptyset, 2)$

$$J(\emptyset,2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$$

$\lambda_1 = -1;$ $\underline{x}_1 =$

$\lambda_2 = -2;$ $-2x_{11} - 2x_{12} = -2x_{12}$

$= \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\underline{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

$-a = -a$

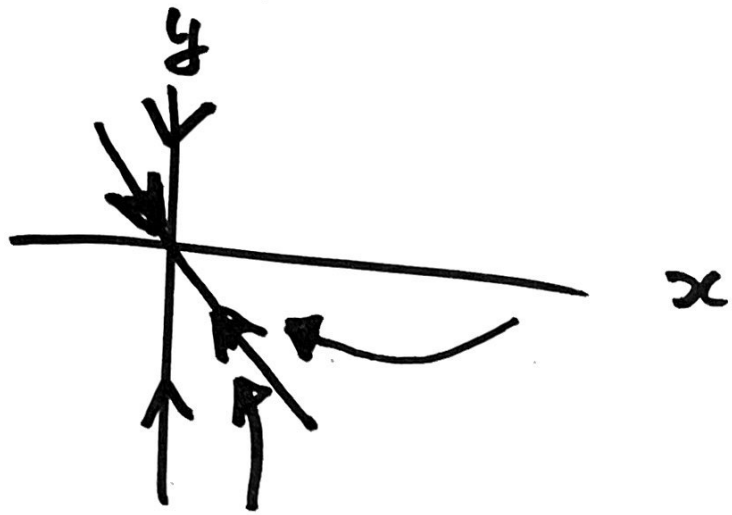
$-2a - 2b = -b$

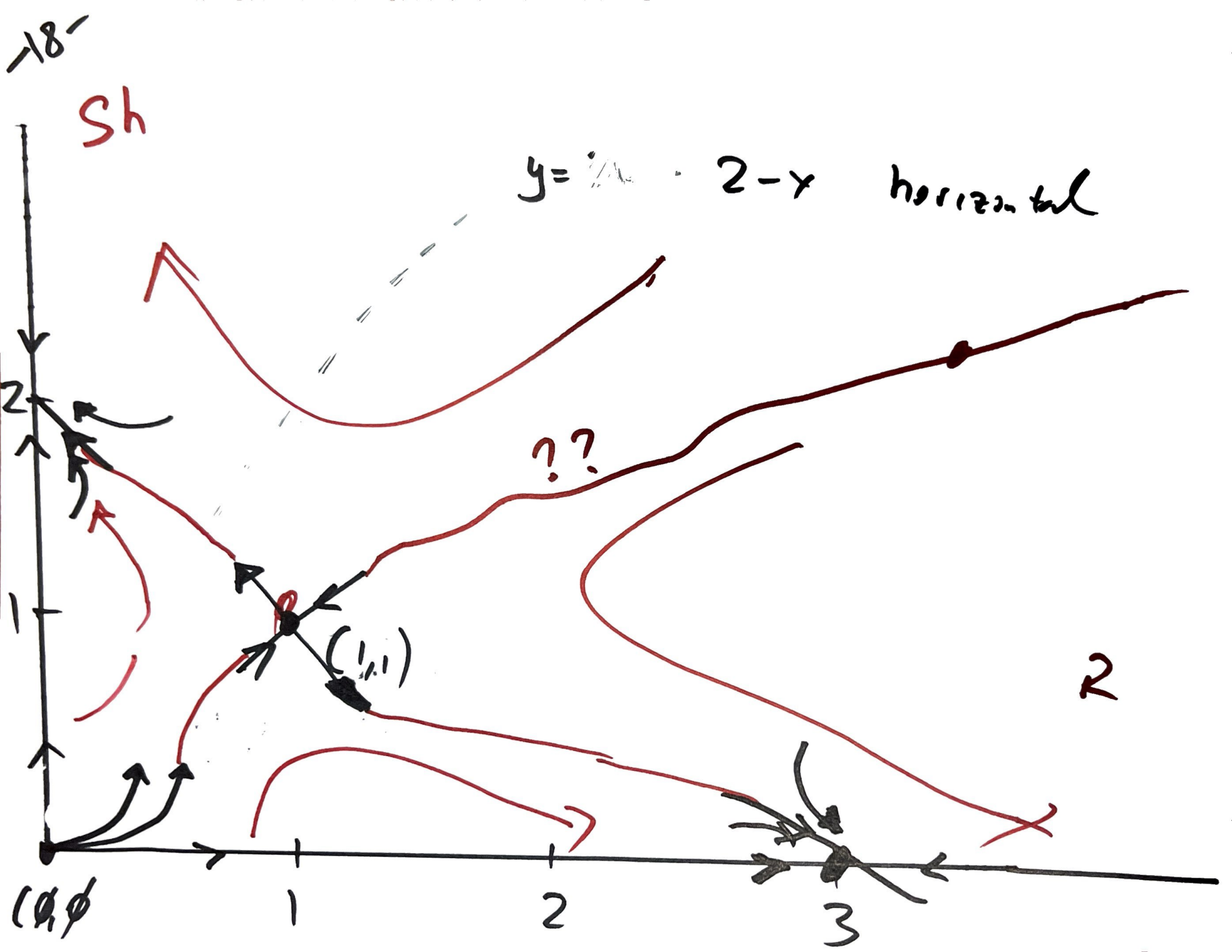
$-2a = b \Rightarrow b = -2$

$a = 1$

-17-

(0,2)




Sh
y= 2-x horizontal
??
(1,1)
2
2
1
(0,0
1
2
3