Conserved quantity

$E(\underline{x})$ such that

$\dot{\underline{x}} = f(\underline{x}) \Leftrightarrow \frac{d}{dt} E(\underline{x}) = 0$

Newton

$F = -\frac{d}{dx} V(x) = m a = m \ddot{x}$

$E(x) = \frac{m\dot{x}^2}{2} + V(x)$ is conserved

$\frac{d}{dt} E(x) = \frac{d}{dt} \left(\frac{m\dot{x}^2}{2} + V(x(t)) \right)$

$= \frac{m}{2} \cdot 2 \cdot \dot{x} \cdot \ddot{x} + \frac{dV}{dx} \cdot \dot{x} =$

$= \dot{x} \left[m\ddot{x} + \frac{dV}{dx} \right] = 0$

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$$E = \left(\frac{d}{dt} x(t)\right)^2 \cdot \frac{m}{2} + V(x)$$

$$\left(\frac{d}{dt} x(t)\right)^2 = \frac{2}{m}\left(-V(x) + E\right)$$

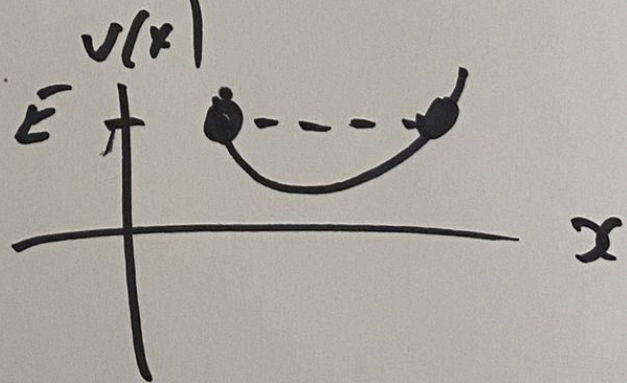
$$\frac{dx}{dt} = \sqrt{\frac{2}{m}\left(-V(x)+E\right)}$$

$$\int dt = \int \frac{dx}{\sqrt{\frac{2}{m}\left(-V(x)+E\right)}}$$

when is $\dot{x} = 0$?

$$E = V(x)$$

↑ to find turning points



-3-

$$V(x) = \frac{x^4}{4} - \frac{x^2}{2}$$

Characterize motion

Find conserved quantities

Find and characterize fixed points

Plot the phase portrait.

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Minima?

$$\frac{dV}{dx} = x^3 - x = x(x^2 - 1)$$

$$x = 0, \quad x = \pm 1$$

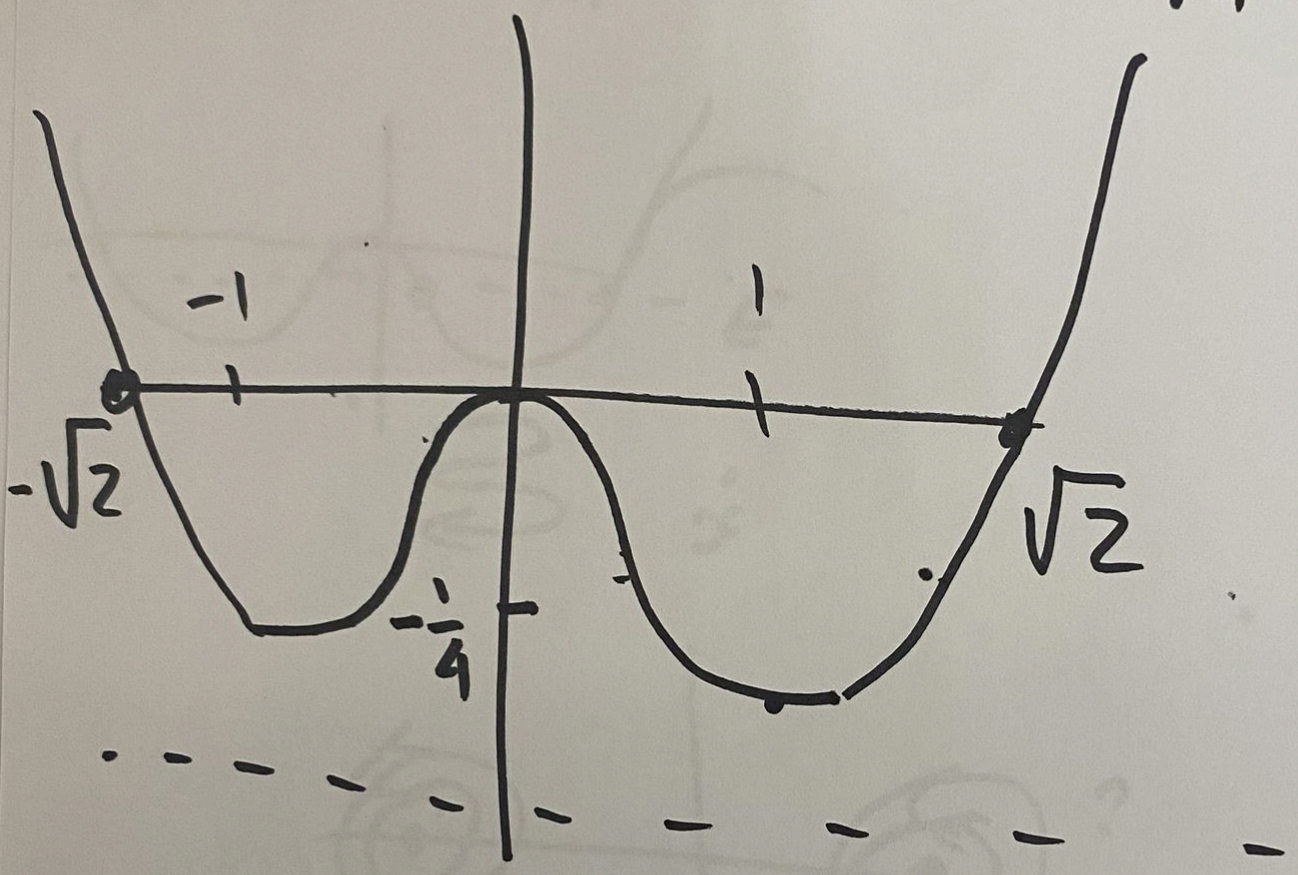
$$V(x) = 0 \Rightarrow \frac{x^4}{4} - \frac{x^2}{2} = 0$$

$$x^4 - 2x^2 = 0$$

$$x^2(x^2 - 2) = 0$$

$$x = 0, \quad x = \pm\sqrt{2}$$

$V(x)$ $\quad V(\pm 1) = -1/4$

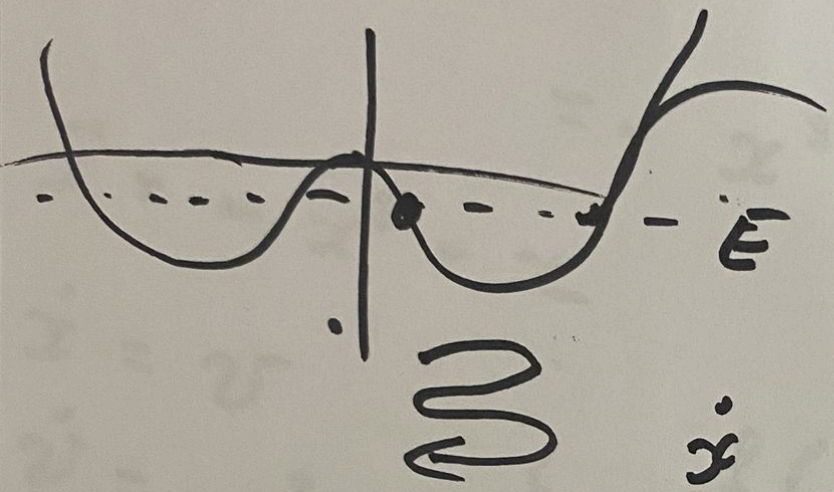


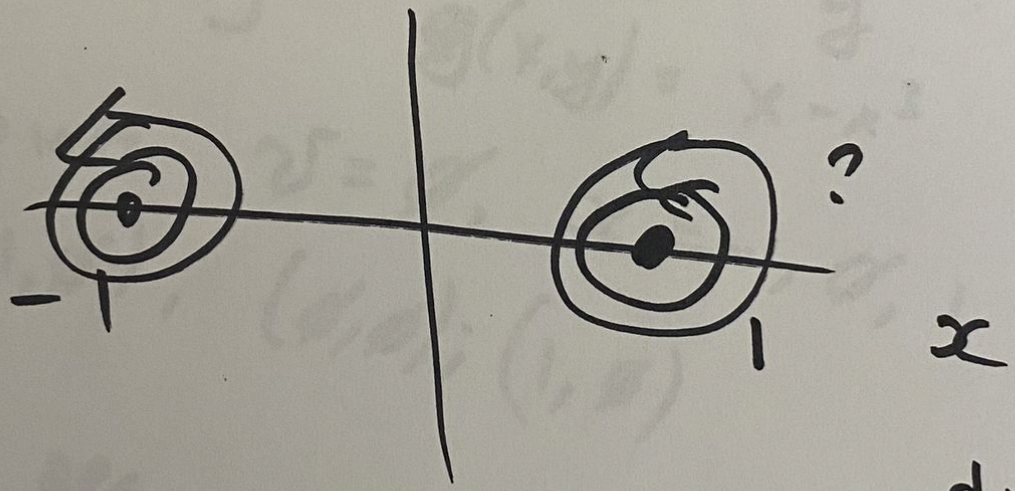
$E \geq -1/4$

$E = -1/4$; $x^* = \pm 1$, $\dot{x} = 0$ are fixed points

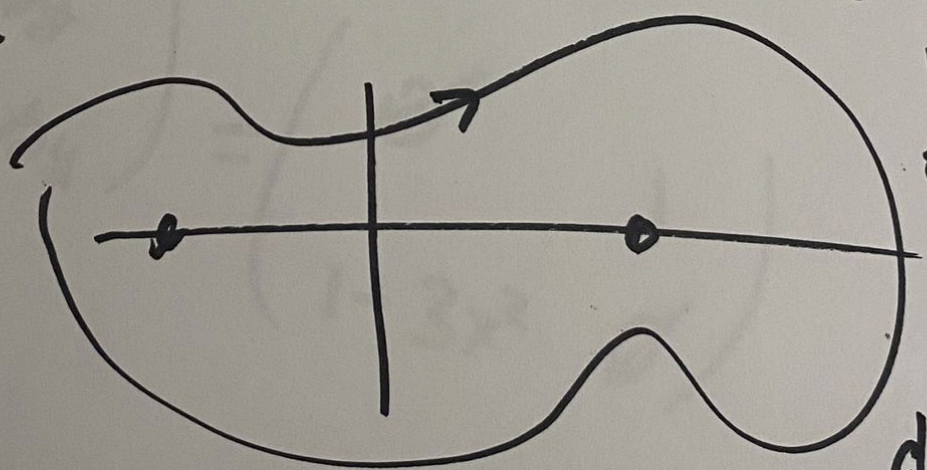
$0 > E \geq -1/4$

$0 > E > -1/4$





$E > 0$



direction is to be determined

-7- $m = 1$ $\quad v \equiv y$

$$m\ddot{x} = F = -\frac{dV}{dx} = -\frac{d}{dx}\left(\frac{x^4}{4} - \frac{x^2}{2}\right)$$

$$= -x^3 + x = x - x^3$$

$$\ddot{x} = -x^3 + x$$

$$\left.\begin{aligned} \dot{x} &= v \\ \dot{v} &= -x^3 + x \end{aligned}\right\} \quad \begin{aligned} f(x,y) &= y \\ g(x,y) &= x - x^3 \end{aligned}$$

Fixed points $v = 0$, $x = -1, 0, 1$

$(-1, 0)$; $(0, 0)$; $(1, 0)$

$$J = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix}$$

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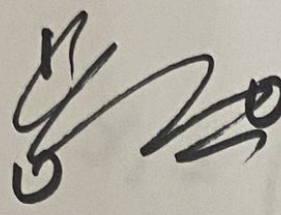
$(x,y) = (0,0)$

$\exists (0,0) =$ 

$T = 0; \quad \Delta = -1 \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4\Delta}}{2} = \frac{1}{2}(\pm\sqrt{4})$$

$$= \pm 1$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$$

$x_{12} = x_{11}$

$\underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$1 \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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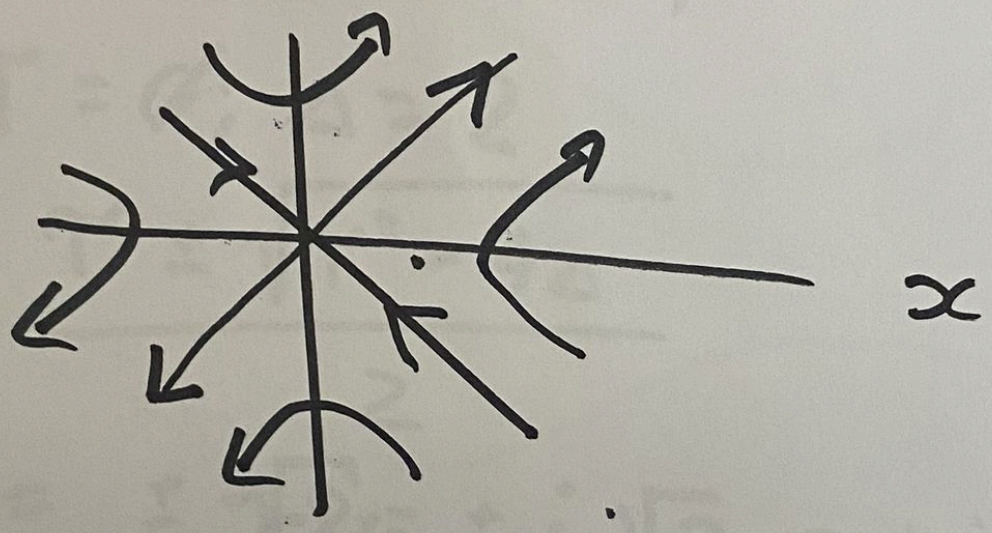
$\lambda_1 = 1; \quad x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1 \quad x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



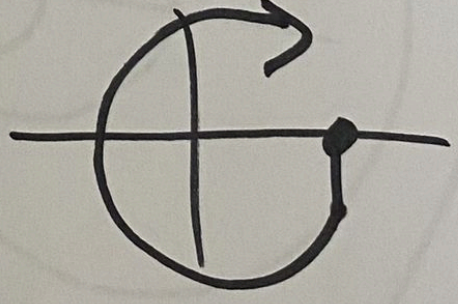
$\underline{x}^* = (1, 0)$

$J(1,0) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$

$T = 0; \quad \Delta = 2$

$\lambda_{1,2} = \dfrac{T \pm \sqrt{T^2 - 4\Delta}}{2}$

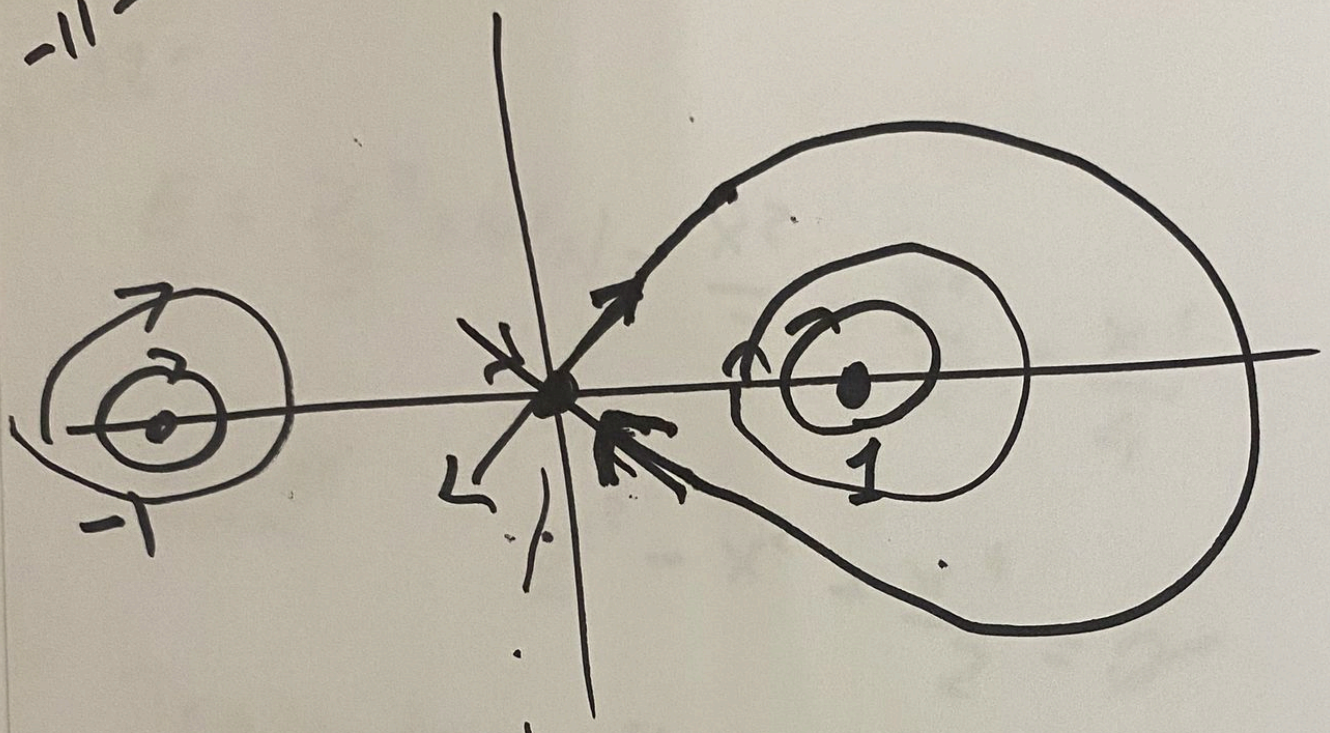
$= \pm \sqrt{-8}/2 = \pm i\sqrt{2} = \pm i\sqrt{2}$

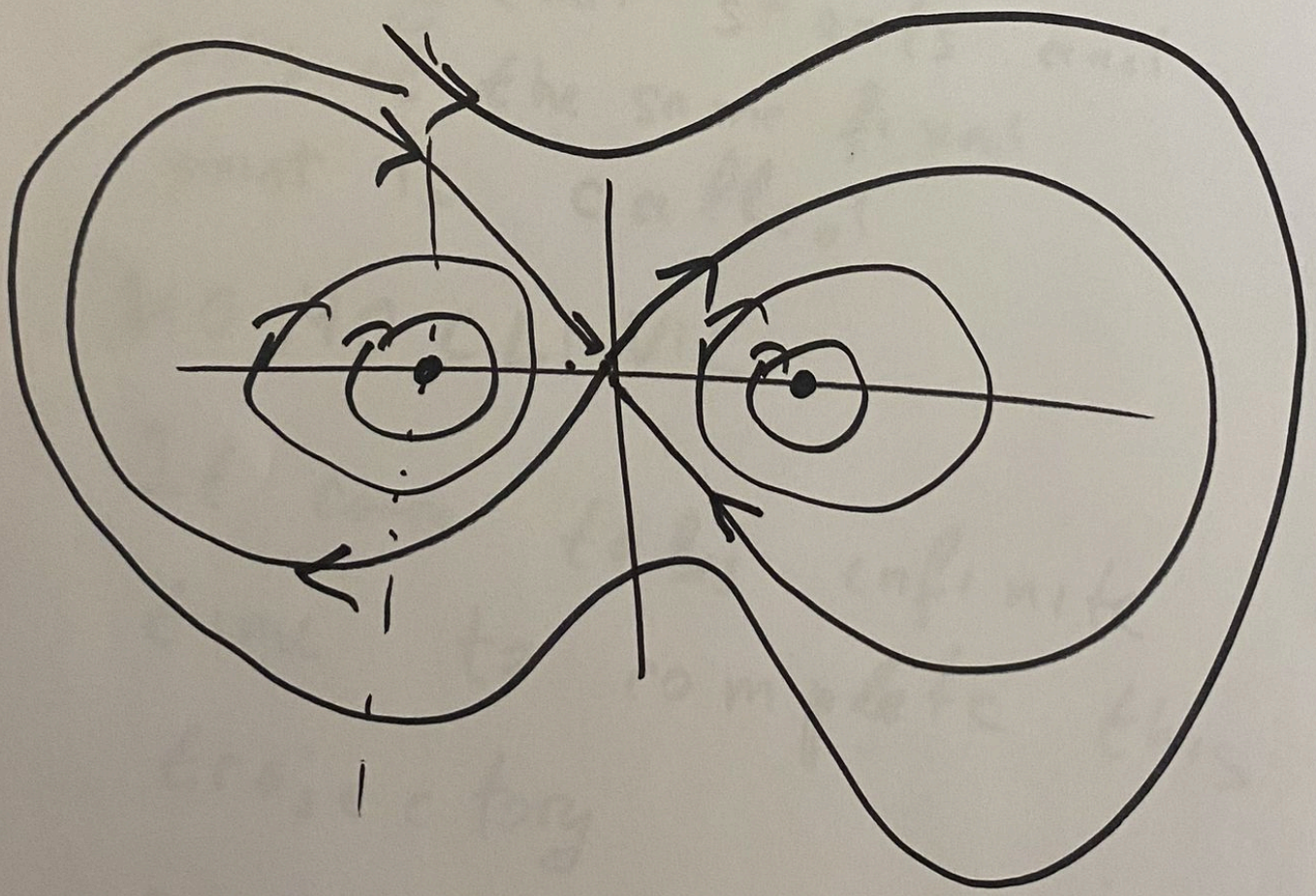


$\underline{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

$x^* = (-1, 0)$ → same


-1
1



$$E = \frac{\dot{x}^2}{2} + V(x) = \frac{\dot{x}^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$$

$E = 0$

$$\frac{\dot{x}^2}{2} - x^2 + \frac{x^4}{2} = 0$$

Trajectory that starts and ends at the same fixed point is called

HOMOCLINIC

It can takes infinite time to complete this trajectory

$$\int dt = \int \frac{dx}{\sqrt{\frac{2}{m}(E - V(x))}}$$

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$$\int dt = \int \frac{dx}{\sqrt{2\left(\frac{x^2}{2} - \frac{x^4}{4}\right)}}$$

$$E = \frac{\dot{x}^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} = 0 = \epsilon$$

$$\ddot{x} = 0 \Rightarrow \frac{x^3}{4} - \frac{x^2}{2} = 0$$

$$x^4 - 2x^2 = 0 \Rightarrow x = 0, \pm\sqrt{2}$$

$$T = \int_0^T dt = \int_0^{\sqrt{2}} \frac{dx}{\sqrt{x^2 - x^4/2}}$$

from 0 to $\sqrt{2}$

14. $E = 1$

Turning points:

$$1 = \frac{\dot{x}^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$$

$\dot{x} = 0$;

$$\frac{x^4}{4} - \frac{x^2}{2} - 1 = 0 \quad ; \quad y = x^2$$

$$\frac{y^2}{4} - \frac{y}{2} - 1 = 0$$

$$y^2 - 2y - 4 = 0$$

$$y_{1,2} = \frac{2 \pm \sqrt{4 + 16}}{2}$$

$$= 1 \pm \sqrt{5}$$

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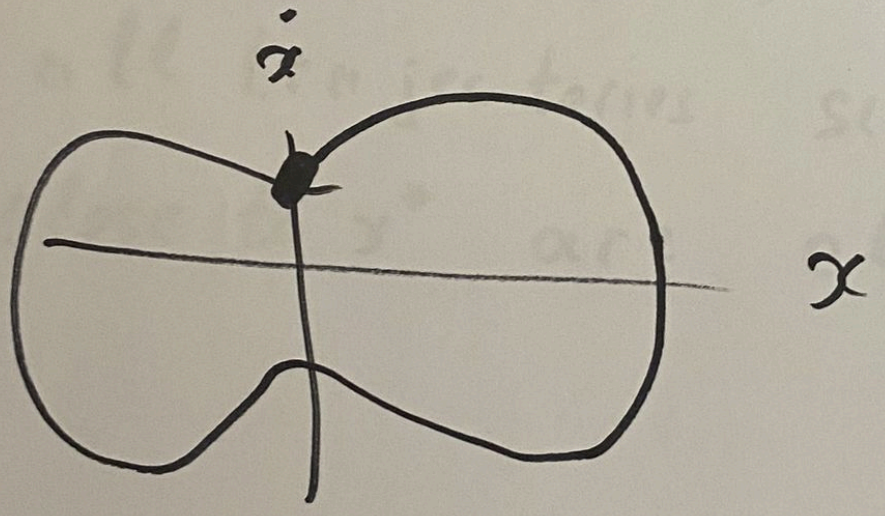
$$x_{1,2,3,4} = \pm\sqrt{1 \pm \sqrt{5}}$$

$$= \pm\sqrt{1 + \sqrt{5}}$$



Time from $\emptyset$ to $\sqrt{1+\sqrt{5}}$

$$T = \int_0^{\sqrt{1+\sqrt{5}}} \frac{dx}{\sqrt{2\left(1 + \frac{x^2}{2} - \frac{x^4}{4}\right)}}$$

Theorem

If $E(x)$ is a conserved quantity, and x^* is a minima of $E(x)$, then all trajectories sufficiently close to x^* are closed (periodic).

Time reversibility.

Some systems are time reversible.

Time reversible is a system where changing direction of time and some other changes leaves equation of motion unchanged (invariant)

Newton's equation is time reversible

$$m\ddot{x} = -\frac{dV}{dx}$$

$$\begin{cases} \dot{x} = u = dx/dt \\ \dot{u} = -\frac{1}{m}\frac{dV(x)}{dx} = du/dt \end{cases}$$

$$t = -\tau$$

$$u = -v \qquad \frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = -\frac{d}{d\tau}$$

$$\frac{dx}{dt} = -\frac{dx}{d\tau} = u = -v \Rightarrow \boxed{\frac{dx}{d\tau} = v}$$

$$\frac{du}{dt} = -\frac{d}{d\tau}u = \boxed{\frac{d}{d\tau}v = -\frac{1}{m}\frac{dV}{dx}}$$