-1-

Time reversibility

$$m\ddot{x} = F = -\frac{dV}{dx}$$

$$v = \dot{x}$$

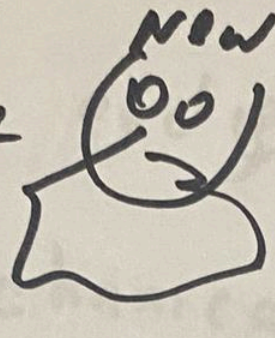
$$\begin{cases} \dot{x} = v \\ \dot{v} = \frac{1}{m}\frac{dV}{dx}(\cdot -1) \end{cases}$$

$$t = -\tau \; ; \quad \frac{d}{dt} = -\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{d\tau}$$

$$u = -v$$

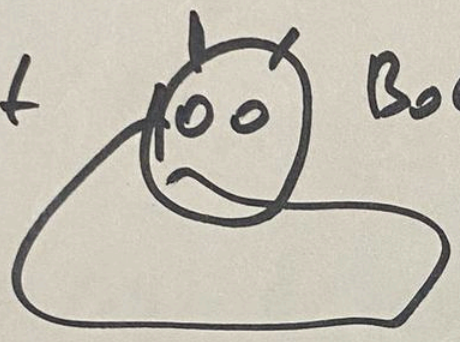
$$\dot{x} = v \Rightarrow \frac{d}{dt}x = -\frac{d}{d\tau}x = v = -u \Rightarrow \boxed{\frac{dx}{d\tau} = u}$$

$$\dot{v} = \frac{dv}{dt} = -\frac{du}{d\tau} = \boxed{\frac{du}{d\tau} = -\frac{dV}{dx}}$$



How come law

law is time reversible,



But Boltzmann time

irreversible.

-3-

write code for
mechanical system

4'

Let $\dot{x}(t) = f(x(t), y(t))$

$\dot{y}(t) = g(x(t), y(t))$ $\Big\}$ (*)

Let $f(x,y)$ be odd in y,

$$f(x,-y) = -f(x,y)$$

Let $g(x,y)$ be even y:

$$g(x,-y) = g(x,y)$$

Then (*) is time reversible.

5/

$t = -\tau$

$y = -\tilde{y}$

$$\frac{d}{dt} x(t) = -\frac{d}{d\tau} x(\tau) = f(x, y) =$$

$$= f(x, -\tilde{y}) = -f(x, \tilde{y})$$

$$\frac{d}{d\tau} x(\tau) = f(x(\tau), \tilde{y}(\tau))$$

$$\dot{y} = \frac{d}{dt} y(t) = -\frac{d}{d\tau} y(\tau) = \frac{d}{d\tau} \tilde{y}(\tau) =$$

$$= g(x(\tau), y(\tau)) = g(x(\tau), -\tilde{y}(\tau))$$

$$= g(x(\tau), y(\tau))$$

$$\frac{d}{d\tau} \tilde{y}(\tau) = g(x(\tau), x(\tau))$$

-6-

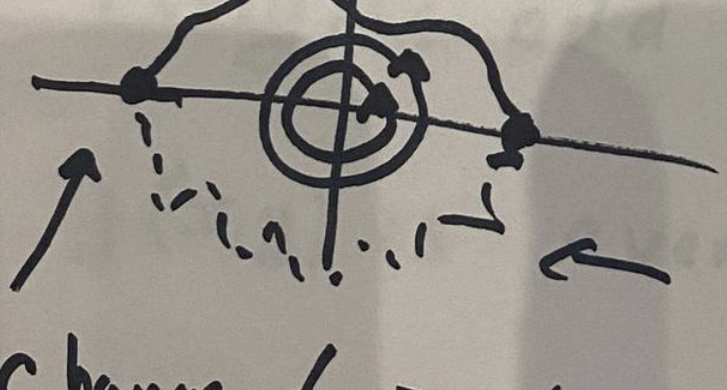
Let Nonlinear center

$$\left.\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}\right\}$$ be time reversible System.

And $x = y = 0$ be a linear center.

Then any trajectory close to $x = y = 0$ is closed and therefore periodic

<u>Proof</u>



change $t \to -t$ and $y \to -y$

Gwin trajectory under time reversal

-7-

$\dot{x} = y - y^3$

$\dot{y} = -x - y^2$

Show origin is a nonlinear center, plot phase center

$f(x,y) = y - y^3 = y(1-y^2)$

$g(x,y) = -x - y^2$

$f(x,y)$ is odd in y: $y - y^3 = -(-y) + (-y)^3$

$g(x,y)$ is even in y:

$$g(x,y) = -x - y^2 = g(x,-y)$$

-6-

$$f(x,y^*) = y^*(1-y^{*2}) = 0$$

$$y^* = 0, \quad y^* = \pm 1$$

$$g(x,y^*) = -x - (y^*)^2$$

If $y^* = 0$, $x^* = 0$;

$y^* = \pm 1$, $x^* = -1$

$(0,0)$; $(-1,1)$; $(-1,-1)$;

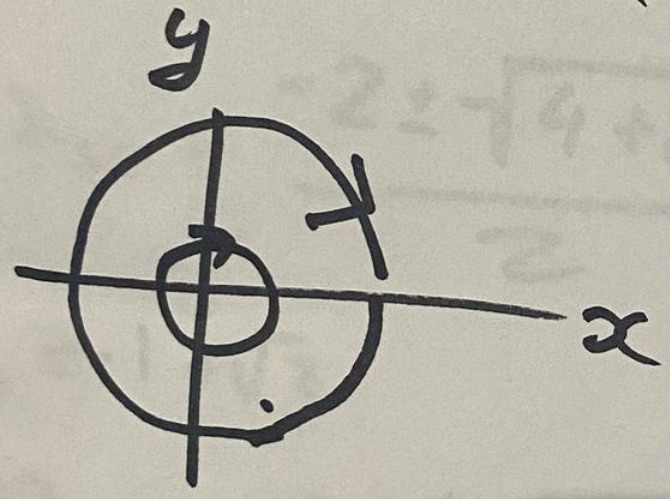
$$J = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix} = \begin{pmatrix} 0 & 1-3y^2 \\ -1 & -2y \end{pmatrix}$$

$$g(\phi,\phi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Linear center,

$\lambda = \pm i$;

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$



Linear center

⇓

time reversible

⇓

NONLINEAR CENTER

-10-

$$J(-1,1) = \begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix}$$

$$-\lambda(-2-\lambda) - 2 = 0$$

$$(\lambda + 2)\lambda - 2 = 0$$

$$\lambda^2 + 2\lambda - 2 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}$$

$$\lambda_1 = -1 + \sqrt{3}$$

$$\begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = (-1+\sqrt{3}) \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$$

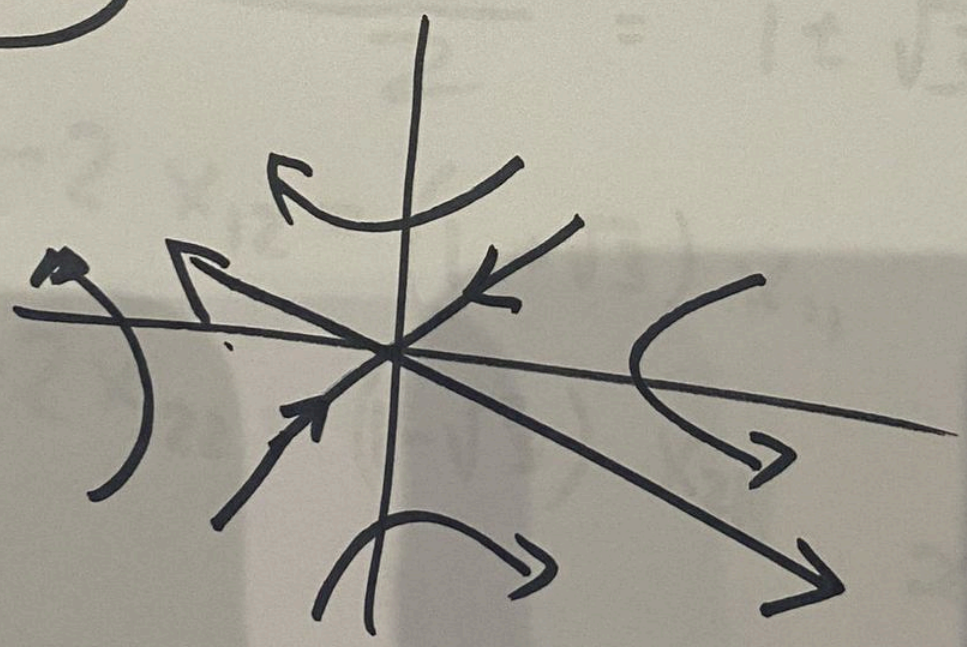
$$-2\,x_{12} = (-1+\sqrt{3})\,x_{11} \Rightarrow x_1 = \begin{pmatrix} -2 \\ -1+\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = (-1-\sqrt{3}) \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$$

$$-2x_{22} = (-1-\sqrt{3})\, x_{21}$$

$$\underline{x}_2 = \begin{pmatrix} 2 \\ 1+\sqrt{3} \end{pmatrix}$$

-1, 1



14 -12-

$\dot{x}$ $(-1,-1)$ $\quad -1,-1$

$$J(-1,-1)=\begin{pmatrix} 0 & -2 \\ -1 & 2 \end{pmatrix}$$

$$-\lambda(+2-\lambda)-2=0$$

$$\lambda(\lambda-2)-2=0$$

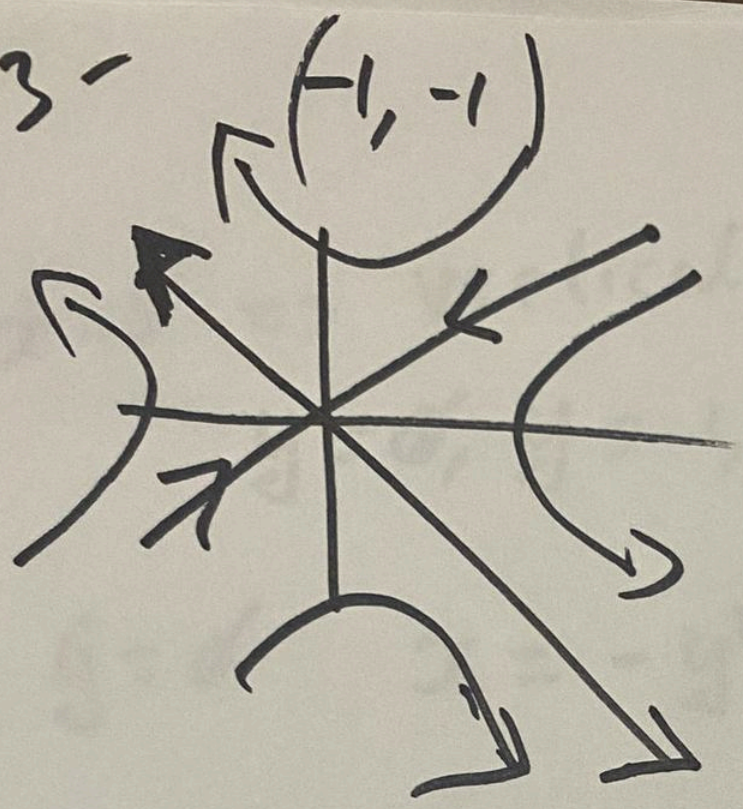
$$\lambda^2-2\lambda-2=0$$

$$\lambda_{12}=\frac{2\pm\sqrt{4+8}}{2}=1\pm\sqrt{3}$$

- $-2x_{12}=(1+\sqrt{3})x_{11}$
- $-2x_{22}=(1-\sqrt{3})x_{21}$

$$\underline{x}_1=\begin{pmatrix} -2 \\ 1+\sqrt{3} \end{pmatrix}$$

$$x_2=\begin{pmatrix} -2 \\ 1-\sqrt{3} \end{pmatrix}$$


(-1, -1)

14

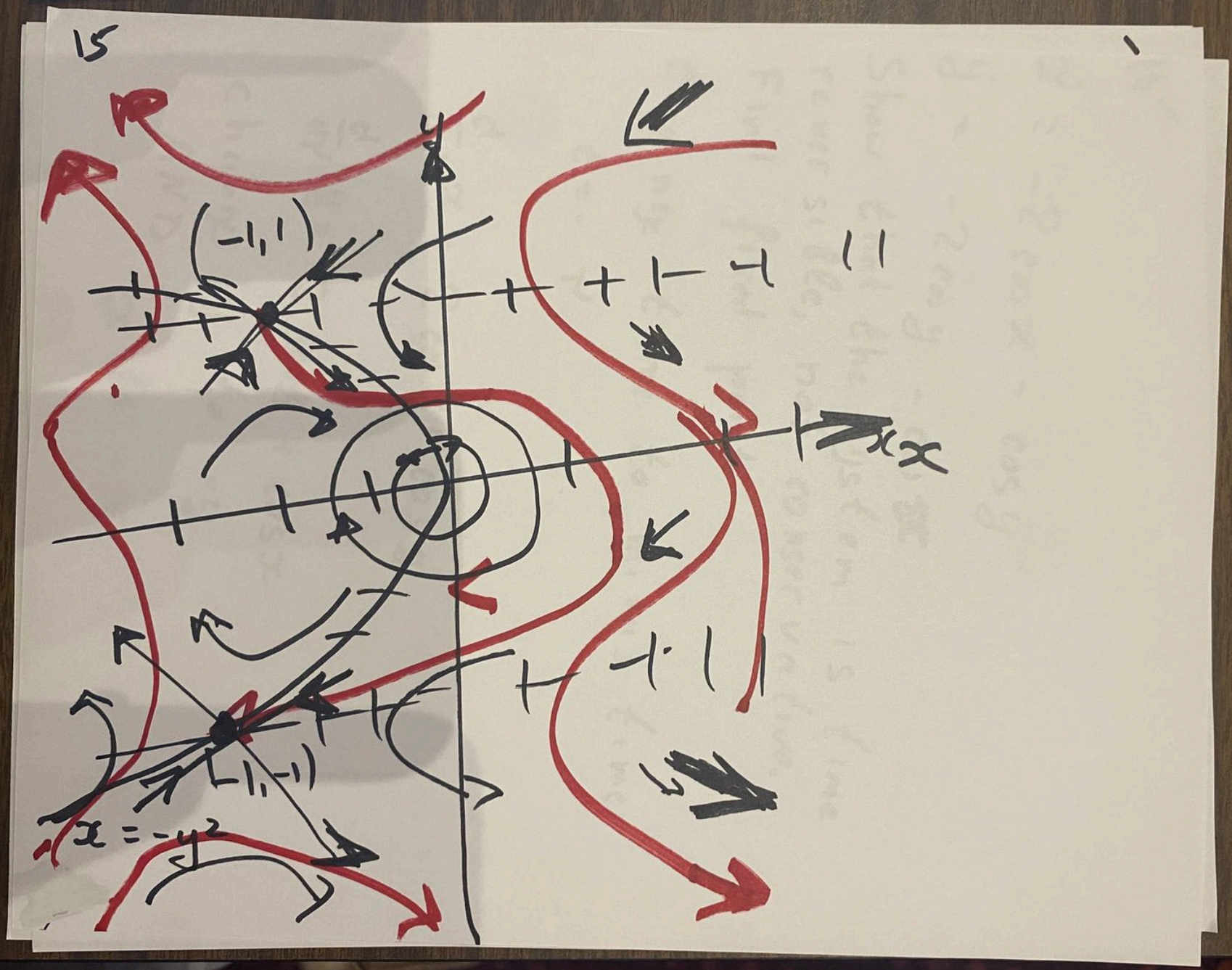
$\dot{x} = 0$ => vertical direction

$y = 0, y = -1, y = 1$ vertical

$\dot{y} = 0$ $x = -y^2$ =>



horizontal


15
(-1, 1)
(-1, -1)
x = -1
y
x

-16-

$\dot{x} = -2\cos x - \cos y$

$\dot{y} = -2\cos y - \cos x$

Show that the system is time reversible, not conservative.

Find fixed points

Change time to minus time

$t = -\tau$

$\frac{d}{d\tau} x = 2\cos x + \cos y$

$\frac{d}{d\tau} y = 2\cos y + \cos x$

change y to $-y$

AND

x to $-x$

-17-

$-2\cos x - \cos y = 0$

$-2\cos y = \cos x = 0$

$\cos y = -2\cos x$

$-2\cos y = 4\cos x = \cos x$

~~$x = \pi n/2; \; n = 1, 2, 3 \ldots$~~

$x = \pi \cdot (2n+1)/2 \; ; \; n = \pm 1, \pm 2, \pm 3$

$x = \frac{\pi}{2}; -\pi/2 \qquad y = \pi(2n+1)/2$

$y = \frac{\pi}{2}; -\pi/2$

-17-

$$J(x,y) = \begin{pmatrix} 2\sin x & \sin y \\ \sin x & 2\sin y \end{pmatrix}$$

$\underline{x}_1^* = (\pi/2, \pi/2)$;

$x_2^* = (-\pi/2, \pi/2)$

$x_3^* = (-\pi/2, -\pi/2)$

$x_4^* = (+\pi/2, -\pi/2)$

$J_1 = J(\pi/2, \pi/2) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$J_2 = J(-\pi/2, \pi/2) = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$

$J_3 = J(-\pi/2, -\pi/2) = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$

$J_4 = J(\pi/2, -\pi/2) = \begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}$

18 $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\lambda_1 = 1$ $\quad x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 3$ $\quad x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$T = 4 \quad \Delta = 3$

$\lambda_{12} = \dfrac{4 \pm \sqrt{16 - 12}}{2}$