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Limiting Cycle

entirely nonlinear

Limiting cycle is an isolated periodic trajectory.

Isolated - no nearby periodic trajectories.



Is there a strategy to find conserved quantity.

~~$\frac{\partial}{\partial t} u(x,t)$~~

$$i \frac{\partial}{\partial t} u(x,t) + \frac{\partial^2}{\partial x^2} u(x,t) + |u(x,t)|^2 u(x,t) = 0$$

Polar coordinates

$\dot{x} = f(x,y)$

$\dot{y} = g(x,y)$

$\rightarrow x = r\cos\varphi$

$y = r\sin\varphi$

$r = (x^2+y^2)^{1/2}$

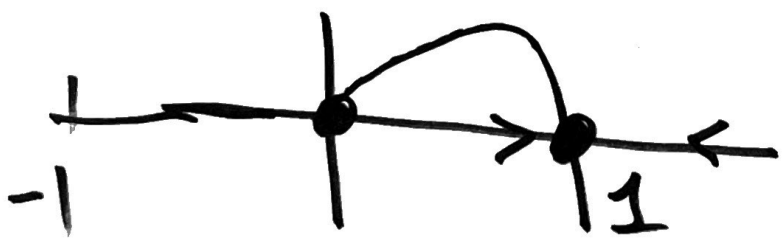
$\varphi = \arctan y/x$

$\dot{\theta} = 1$

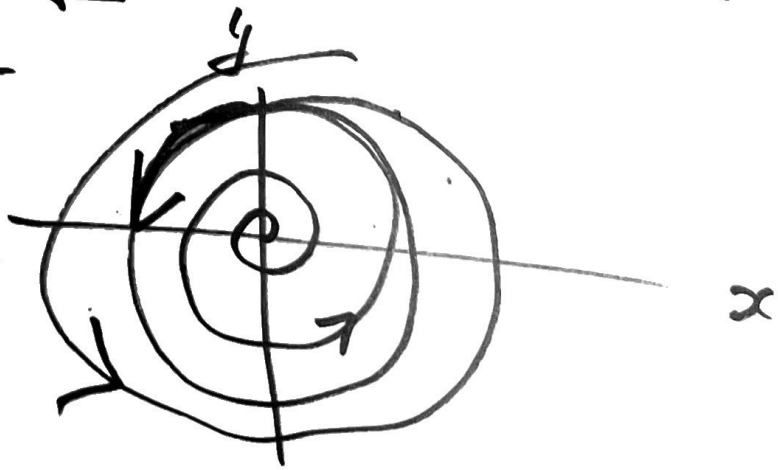
$\dot{r} = r(1-r^2)$

$r = 1$ - stable

$r = 0$ unstable



unphysical



Example of limit cycle

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$$\frac{d}{dt} x(t) = \frac{d}{dt}\left(r(t)\cos\varphi(t)\right)$$

$$= \dot{r}\cos\varphi - r\sin\varphi \cdot \dot{\varphi}$$

$$= r(1-r^2)\cos\varphi - r\sin\varphi \quad (=)$$

$$= \frac{\sqrt{x^2+y^2}}{1-x^2-y^2}\,\frac{x}{\sqrt{x^2+y^2}} - \frac{\sqrt{x^2+y^2}\,y}{\sqrt{x^2+y^2}}$$

~~$\dot{x} = \frac{x}{1-x^2-y^2} - y$~~

$$= -\sqrt{x^2+y^2}\left(1-x^2-y^2\right)\frac{x}{\sqrt{x^2+y^2}} - \frac{\sqrt{x^2+y^2}\,y}{\sqrt{x^2+y^2}}$$

$$= \left(1-x^2-y^2\right)x - y =$$

$$= x - x(x^2+y^2) - y = (x-y) - x(x^2+y^2)$$

$$\dot{y} = \frac{d}{dt}(r \sin\varphi)$$

$$= \dot{r} \sin\varphi + r \cos\varphi$$

$$= r(1-r^2)\frac{y}{r} + r \cdot x/r$$

$$= (1-x^2-y^2)y + x$$

$$= (x+y) - y(x^2+y^2) = \dot{y}$$

$$(x-y) - x(x^2+y^2) = \dot{x}$$

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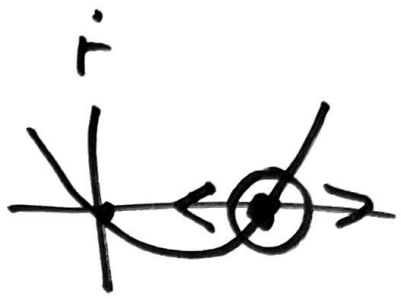
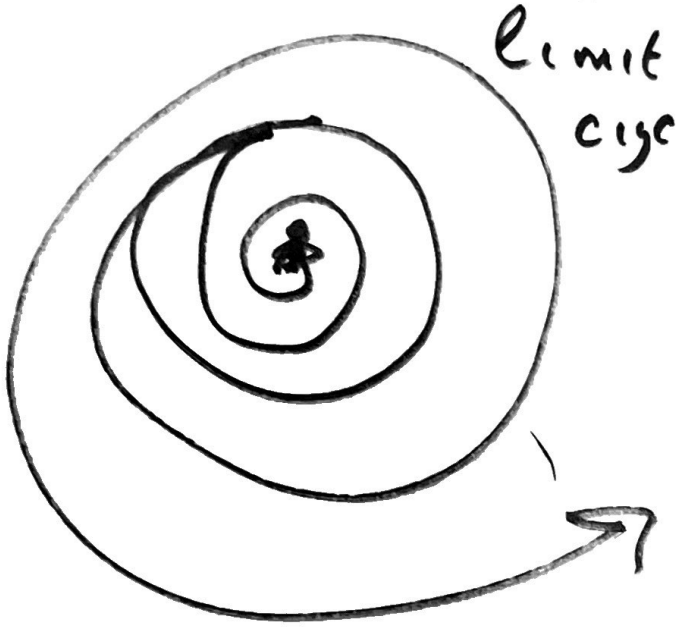
$\dot{r} = r(r^2 - 1)$

$\dot{\theta} = 1$





$\dot{r} = r(1 - r^2)$ - stable limit cycle

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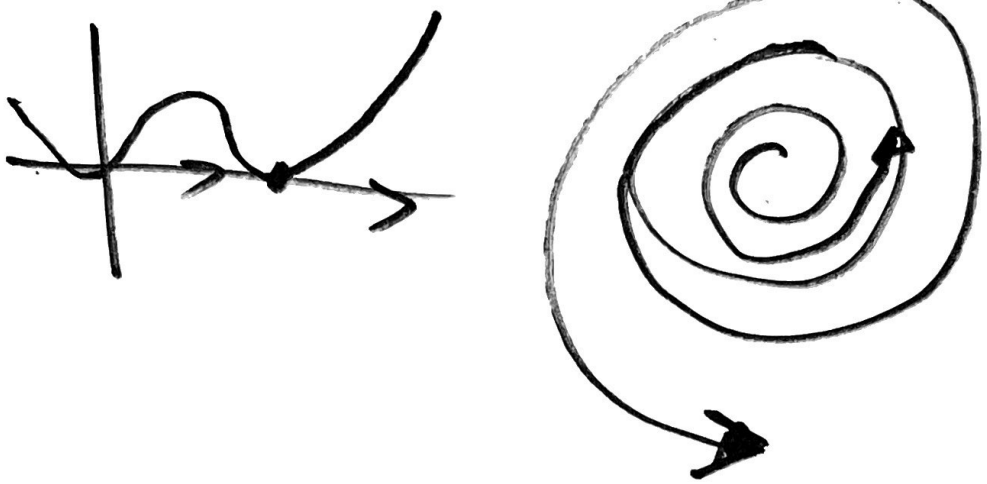
$$\dot{\theta} = 1$$

$$\dot{r} = r(1 - r^2)^2$$



Gradient system

$$\frac{d}{dt}x(t) = f(x(t),y(t)) = \frac{\partial}{\partial x}\varphi(x(t),y(t))$$

$$\frac{d}{dt}y(t) = g(x(t),y(t)) = \frac{\partial}{\partial y}\varphi(x(t),y(t))$$

Th Gradient systems do not have limit cycles.

If the system is gradient system, there is no limit cycle.

Proof

Let there be a limit cycle

$\underline{x}_L(t)$ - limit cycle trajectory

$$\dot{\underline{x}}_L(t) = \underline{x}_L(t+T)$$

$$\varphi(t) = \varphi(t+T)$$

$$\Delta\varphi = \varphi(t+T) - \varphi(t) = 0$$

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$$\Delta\varphi = \varphi(T+t) - \varphi(t)$$

$$= \int_{T=t}^{T+t} \dot{\varphi}(T)\,dT$$

$$= \int_{T=t}^{\tau=T+t} d\tau\ \dot{\varphi}(\tau)$$

$$= \int_{T=t}^{T=t+\tau} d\tau\ \dot{\varphi}(x(T), y_T)$$

$$= \int_{T=t}^{\tau=t+T} d\sigma \left(\frac{\partial\varphi}{\partial x}\cdot\dot{x} + \frac{\partial\varphi}{\partial y}\cdot\dot{y}\right)$$

$$= \int_{T=t}^{t+T} d\tau \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] \geq 0$$

$\dot{x} = \dot{y} = 0$

10. Show that the system

$$\dot{x} = \sin y$$
$$\dot{y} = x \cos y$$

has no limiting cycles.

Proof

Is there a potential?

$$\dot{x} = \sin y = \frac{\partial}{\partial x}\varphi(x,y)$$

$$\varphi(x,y) = \int \sin y \, dx$$

$$= x \sin y + F(y)$$

$$\dot{y} = x \cos y = \frac{\partial \varphi}{\partial y} \Rightarrow$$

$$\varphi(x,y) = \int x \cos y \ldots dy =$$

$$= x \sin y + G(x)$$

$$F(y) = G(x) = C; \quad \varphi(x,y) = x \sin y$$

$$\frac{\partial}{\partial y}\left[\dot{x} = \frac{\partial \varphi}{\partial x}\right]$$

$$\frac{\partial}{\partial x}\left[\dot{y} = \frac{\partial \varphi}{\partial y}\right]$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial g(x,y)}{\partial x}$$

$$f(x,y) = \sin y$$

$$g(x,y) = x \cos y$$

$$\frac{\partial f}{\partial y} = \cos y$$

$$\parallel$$

$$\frac{\partial g}{\partial x} = \cos y$$

Consider nonlinear oscillator

$$\ddot{x} + (\dot{x})^3 + x = 0 \Rightarrow \ddot{x} + x = -(\dot{x})^3$$

Show, there is no limiting cycle.

Proof

$$E(x, \dot{x}) = \frac{x^2 + \dot{x}^2}{2}$$

$E(x, \dot{x}) > 0 \quad \forall \; x \neq 0, \; \dot{x} \neq 0$

$E(x = \dot{x} = 0) = 0$

$d/dt \; E(x, \dot{x}) < 0$

Let there be limiting cycle $x(t)$

Then $E(x(t+T), \dot{x}(t+T)) = E(x(t), \dot{x}(t))$

$$\Delta E = E(x(t+T), \dot{x}(t+T)) - E(x(t), \dot{x}(t)) = \int_{T=t}^{T=t+T} dT \frac{d}{dt} E(x(t), \dot{x}(t)) = \int dT (\dot{x})^4 \geq 0$$

$$\frac{d}{dt} E(x, \dot{x}) = \frac{\partial E}{\partial x} \cdot \dot{x} + \frac{\partial E}{\partial \dot{x}} \ddot{x} = x \cdot \dot{x} + \dot{x} \ddot{x} = \dot{x}(x + \ddot{x}) = -(\dot{x})^4$$

$$\frac{d}{dt} F(x(t), y(t)) = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt}$$

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Lapunov function

1857 - 1918, Russia

Consider

$$\dot{\underline{x}} = \underline{f}(\underline{x}(t))$$

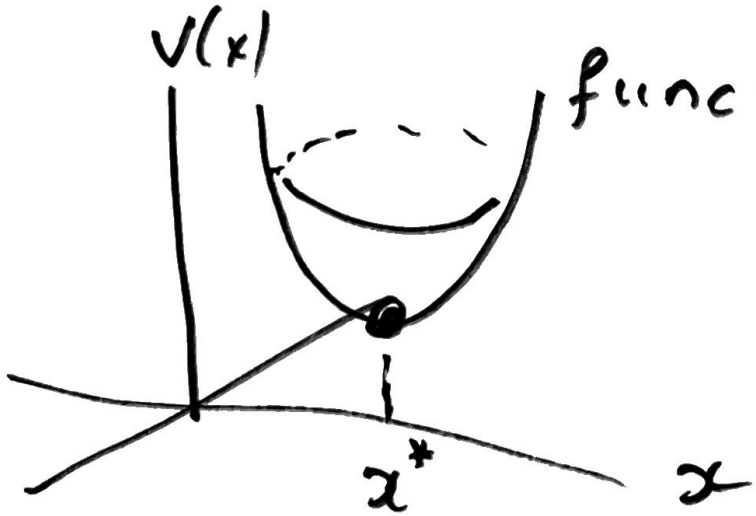
Let $V(x)$ be such that

(1) $V(\underline{x}) > 0$ for all values of x except x^*, with

$V(x^*) = 0$ with $\underline{f}(\underline{x}^*) = 0$

(2) $\frac{d}{dt} V(\underline{x}(t)) < 0$

Then $V(x)$ is a Lapunov's function.



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$\dot{x} = \sin y$ $\quad y = \pi n$

$\dot{y} = x \cos y$ $\quad x = 0$

$(0, \pi n)$, $n = 0, \pm 1, \pm 2 \ldots$

$$J = \begin{pmatrix} 0 & \cos y \\ \cos y & -x \sin y \end{pmatrix}$$

$$J = \begin{pmatrix} 0 & (-1)^n \\ (-1)^n & 0 \end{pmatrix}$$

$n = 1$ $\quad J = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ $\quad \lambda^2 = 1$

$\lambda = \pm 1$

$\lambda_1 = 1, \quad x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = -1 \quad x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

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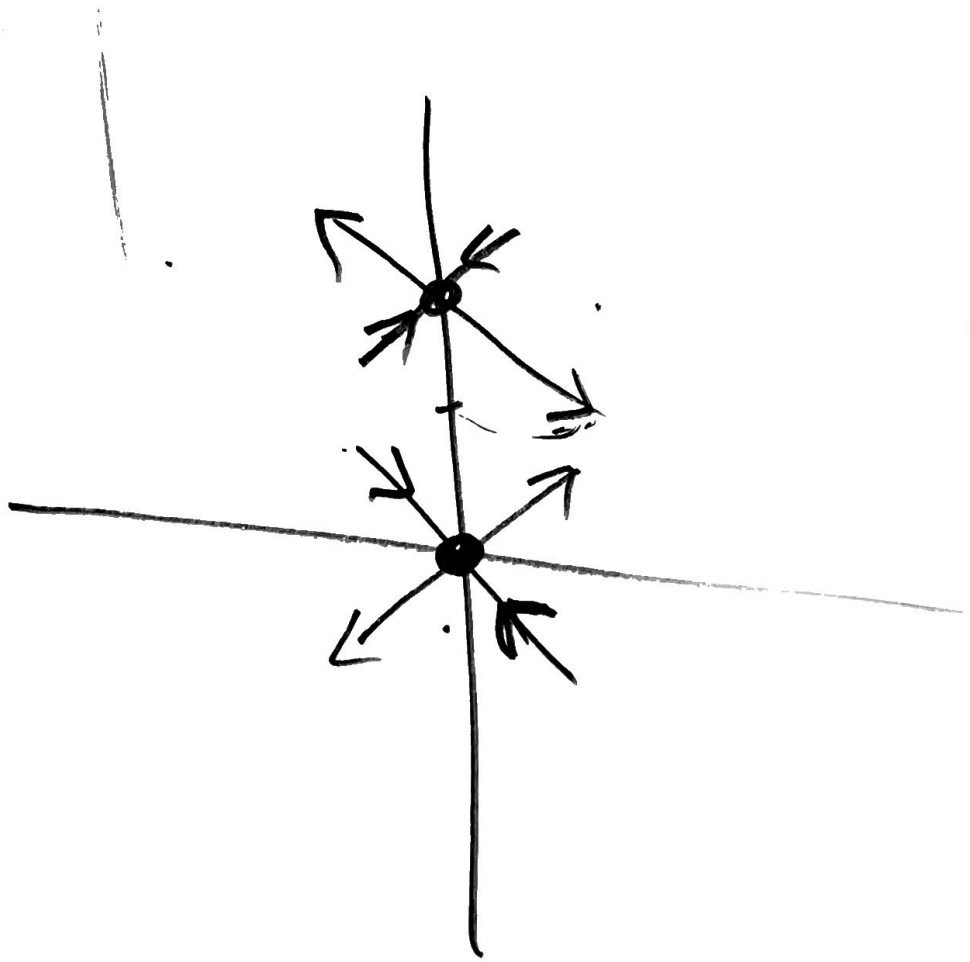
$\underline{n = \emptyset}$

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\lambda_1 = 1 \qquad x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1 \qquad x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



$$\dot{x} = -x + 4y$$

$$\dot{y} = -x - y^3$$

$V(x, y) = x^2 + \alpha y^2$, Find α

$$\frac{d}{dt} V(x,y) = 2x(-x+4y) + 2\alpha y(-x-y^3)$$

$$= \frac{\partial V}{\partial x}\cdot\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= -2x^2 + 8xy - 2\alpha xy - 2\alpha y^4$$

$$= -2x^2 + (8 - 2\alpha)xy - 2\alpha y^4$$

$$= -2(x^2 - (\alpha - 4)xy + \alpha y^4)$$