1) No closed

$\dot{x} = -x + 4y$ Show, there is no closed orbits

$\dot{y} = -x - y^3$

$\mathcal{L} = x^2 + a \cdot y^2$ a is to be found

① $\dot{\mathcal{L}} = 2x\dot{x} + 2ay\dot{y}$

② $= 2x(-x + 4y) + 2ay(-x - y^3)$

③ $= -2x^2 + 8xy - 2axy - 2ay^4$ (2×4)

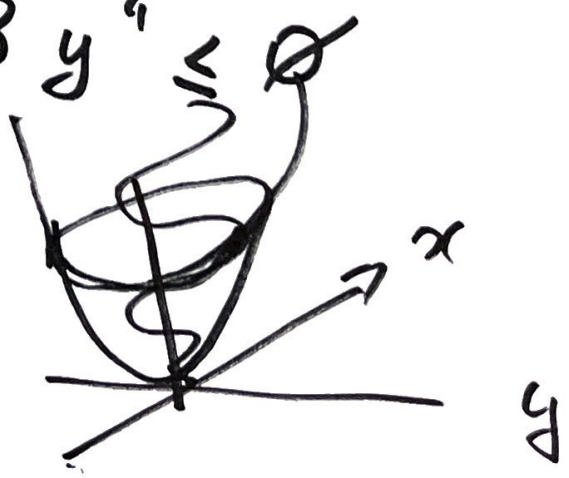
④ $= -2x^2 - 2xy(a - 4) - 2ay^4$

⑤ $a = 4$:

⑥ $\dot{\mathcal{L}} = -2x^2 - 8y^4 \leq 0$

⑦ $\mathcal{L}(x,y) = x^2 + 4y^2$

⑧ $\frac{d}{dt}\ \mathcal{L}(x,y) \leq 0$



2 Let $\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t))$;

Let $\underline{f}(\underline{x})$ be continiously differentiable.

If there exists a continiously differentiable function $\underline{g}(\underline{x})$ such that

$\nabla \cdot (\underline{g}(x) \cdot \dot{x})$ has same sign, then there is no closed orbits.

-3-

$$\dot{x} = x(2-x-y)$$

$$\dot{y} = y(4x-x^2-3)$$

$$g(x,y) = \frac{1}{xy}$$

$$\nabla \cdot (g(x,y)\,\underline{x}(t)) =$$

$$= \nabla \cdot \left(\frac{1}{xy} \begin{pmatrix} x(2-x-y) \\ y(4x-x^2-3) \end{pmatrix} \right)$$

$$= \nabla \cdot \begin{pmatrix} \frac{1}{y}(2-x-y) \\ \frac{1}{x}(4x-x^2-3) \end{pmatrix}$$

4

$$\nabla \cdot (g \dot{\underline{x}}) =$$

$$= -\frac{1}{y}$$

NO closed orbits $y > 0$

Poincare - Bendixson Theorem

1854-1912 1861-1935

(1) $\mathcal{R}$ is closed and bounded set of $\mathbb{R}^2$

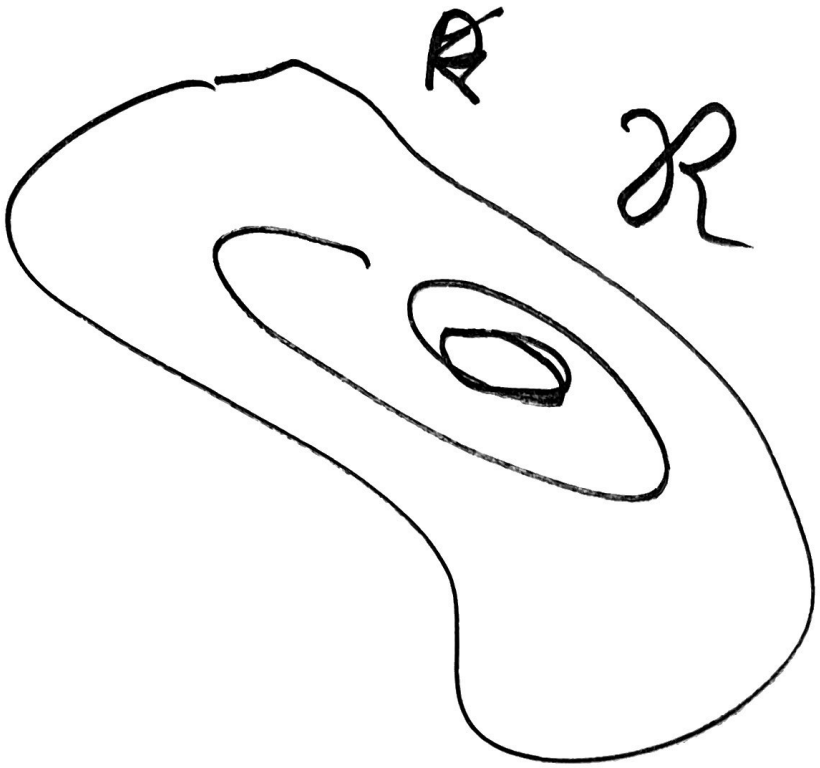
(2) $\dot{\underline{x}} = \underline{f}(\underline{x})$, $\underline{f}(\underline{x})$ is continuously differentiable on a set containing $\mathcal{R}$

(3) $\mathcal{R}$ does not contain any FP

(4) C is a trajectory totally contained in $\mathcal{R}$

Then C is either ↗ closed orbit
↘ C travels towards closed orbit

-6-



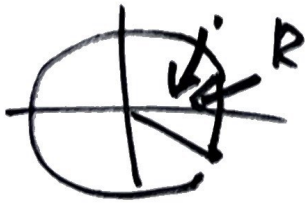
$$\dot{\theta}(t) = 1$$

$$\dot{r}(t) = r(t)(1 - r^2(t)) + \varepsilon r(t)\cos\theta(t)$$

Show that are closed orbits if $\varepsilon > 0$ is sufficiently small.

Find such value of R^+ so that

if $r(t) = R^+$ then $\dot{r}(t) < 0$



$\dot{r}(t) < 0$ "worst" θ is such that $\cos\theta = 1$

$-1 \leq \cos\theta \leq 1$

-8-

$$\dot{r} = r(1-r^2) + \mu r$$

$$= r(\mu + 1 - r^2) < 0$$

$$r > \sqrt{\mu+1}$$

IF $r > R^+ = \sqrt{\mu+1}$ then

$$\dot{r} < 0$$

IF $r < R^-$; $\dot{r} > 0$

$$\dot{r} = r(1-r^2) + \mu r \cos\theta$$

$\cos\theta = -1$;

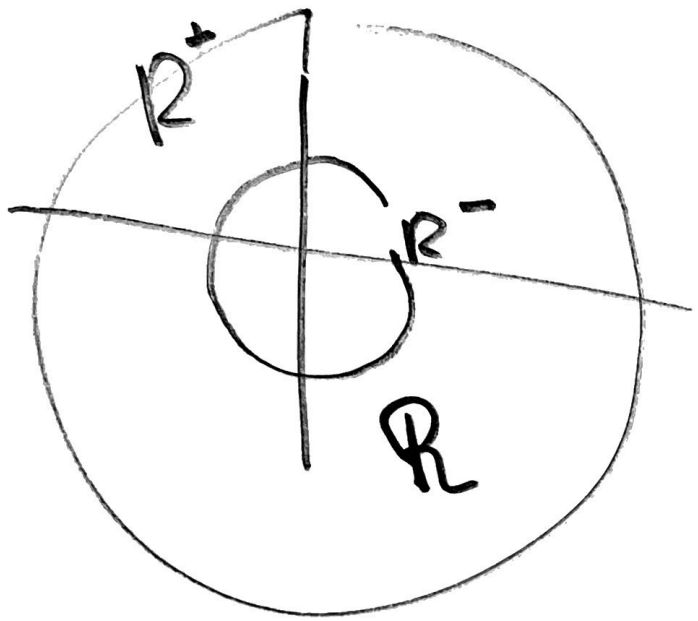
$$\dot{r} = r_0(1-r^2) - \mu r =$$

$$= r(1-\mu-r^2)$$

$$R^- = \sqrt{1-\mu}$$

If $r > R^+ = \sqrt{1+s^n}$, then $\dot{r} < 0$

If $r < R^- = \sqrt{1-s^n}$ then $\dot{r} > 0$





Trapping region

Then there is a limiting cycle

Lienard Systems

$\ddot{x}(t) + f(x)\dot{x} + g(x) = 0$

nonlinear oscillator

$g(x)$ - restoring force, $g(x) > 0$ if $x > 0$

$f(x)$ - nonlinear damping

1. $f(x), g(x)$ are continuously differentiable
2. $g(x)$ is odd, $g(-x) = -g(x)$
3. $g(x) > 0$ if $x > 0$
4. $f(x)$ is even, $f(x) = f(-x)$
5. The function $F(x) = \int_0^x f(u)du$ has one positive zero at $x = a > 0$, and $F(x) < 0$, $x < a$

nondecreasing for $x > a$, $\lim_{x \to \infty} F(x) = \infty$

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Van-der-Paul oscillator

VAN DER PAUL

$$\ddot{x} + \dot{x}(x^2-1)\mu + x = 0$$

$\mu > 0$

1. $f(x), g(x)$ - continuously diff.
2. $g(x) = x$ is odd
3. $g(x) > 0$, $x > 0$, $x > 0$, if $x > 0$
4. $f(x) = x^2 - 1$ is even
5. $F(x) = \int_0^x f(x)dx = \int_0^x \mu(x^2-1)dx =$

$$= \mu\left(\frac{x^3}{3} - x\right) = \mu x\left(\frac{x^2}{3} - 1\right)$$



Perturbation theory.

$b > 0$

$$\varepsilon x^2 + bx + c = 0, \quad \varepsilon << 1$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4\varepsilon c}}{2\varepsilon}$$

$$= -\frac{b}{2\varepsilon} \pm \frac{|b|}{2\varepsilon}\sqrt{1 - \frac{4\varepsilon c}{b^2}}$$

If ε is tiny

$$x_1 = -\frac{b}{2\varepsilon} - \frac{|b|}{2\varepsilon} \approx -\frac{b}{\varepsilon}$$

$$x = -\frac{b}{2\varepsilon} + \frac{b}{2\varepsilon}\sqrt{1 - \frac{4\varepsilon c}{b^2}}$$

14 $\quad \varepsilon \ll 1$

$$\sqrt{1+\varepsilon} \approx 1 + \varepsilon/2$$

$$x = -\frac{\ell}{2\varepsilon} + \frac{\ell}{2\varepsilon}\left(1 - 2\varepsilon c/\ell^2\right)$$

$$= -\frac{\ell}{2\varepsilon} \cdot \frac{2\varepsilon c}{\ell^2}$$

$$= -\frac{c}{\ell}$$